Degreewise n-projective model structures on chain complexes

2012 CMS Winter Meeting
December 7 - 10, 2012
Fairmont Queen Elizabeth, Montreal

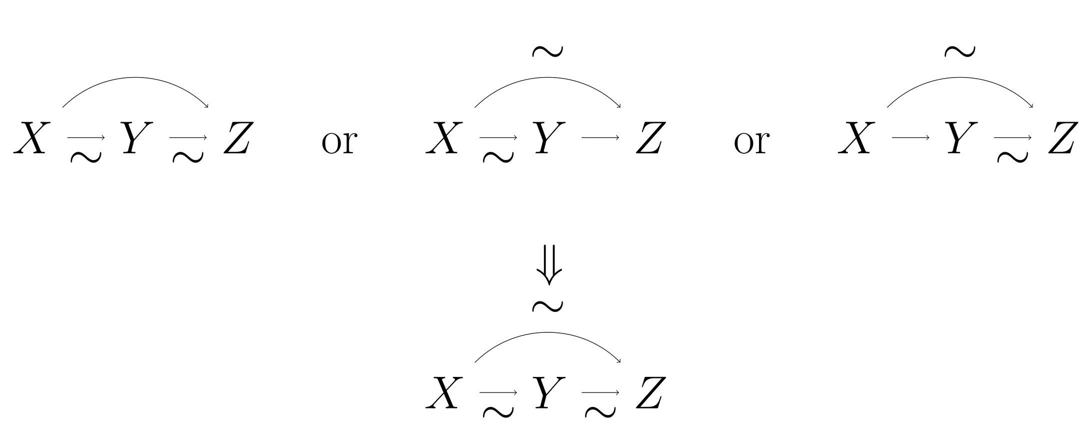
Marco PÉREZ.
Département de Mathématiques. Université du Québec à Montréal.

UQÀM
Centre interuniversitaire de recherches en géométrie et topologie

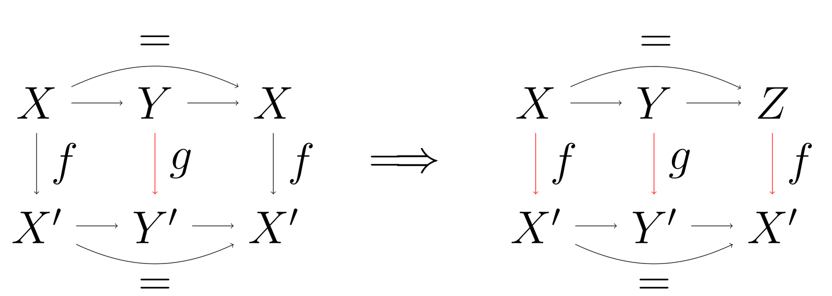
Model Categories

- Let $\mathcal{A}$ be a bicomplete abelian category with enough projective and injective objects. A *model structure* on $\mathcal{A}$ is formed by three classes of morphisms $\mathcal{C}of$, $\mathcal{F}ib$ and $\mathcal{W}$ called **cofibrations**, **fibrations** and **weak equivalences**, respectively, satisfying:

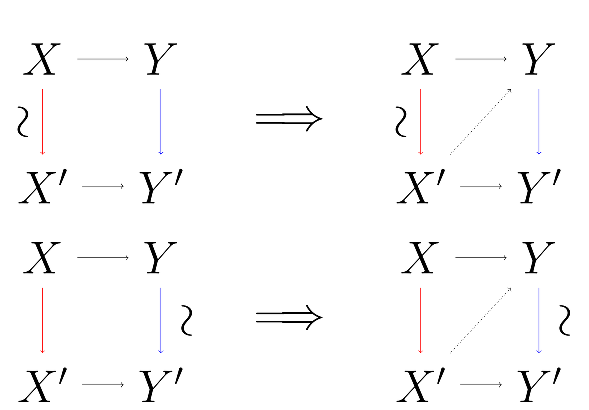
(a) **3 for 2 for weak equivalences**



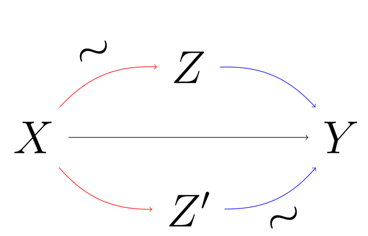
(b) **$\mathcal{C}of$, $\mathcal{F}ib$ and $\mathcal{W}$ are closed under retractions**



(c) **Left lifting property**



(d) **Factorization property**



- An object X is **cofibrant** if $0 \longrightarrow X$ is a cofibration; **fibrant** if $X \longrightarrow 0$ is a fibration; and **trivial** if $0 \xrightarrow{\sim} X$ is a weak equivalence.
- A model structure $(\mathcal{C}of, \mathcal{F}ib, \mathcal{W})$ is *abelian* if
  - (a) $f \in \mathcal{C}of$ iff f is monic with cofibrant cokernel.
  - (b) $g \in \mathcal{F}ib$ iff g is epic with fibrant kernel.

Cotorsion Pairs

- Two classes $\mathcal{C}, \mathcal{F}$ in $\mathrm{Ob}(\mathcal{A})$ form a *cotorsion pair* $(\mathcal{C}, \mathcal{F})$ if:

$$\mathcal{C} = {}^{\perp}\mathcal{F} = \{X : \mathrm{Ext}^1(X, F) = 0 \ \forall\ F \in \mathcal{F}\}$$
$$\mathcal{F} = \mathcal{C}^{\perp} = \{Y : \mathrm{Ext}^1(C, Y) = 0 \ \forall\ C \in \mathcal{C}\}$$

- A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is *complete* if for every object X there are short exact sequences

$$0 \longrightarrow F \longrightarrow C \longrightarrow X \longrightarrow 0$$
$$0 \longrightarrow X \longrightarrow F' \longrightarrow C' \longrightarrow 0$$

- A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is *hereditary* if: $\mathrm{Ext}^i(C, F) = 0$, $\forall\ C \in \mathcal{C}$, $\forall\ F \in \mathcal{F}$, and $\forall\ i > 0$.
- **Example:** *n-projective modules* ($n > 0$):

$$\mathcal{P}_n = \{M \in R\text{-Mod} : \mathrm{pd}(M) \leq n\}$$

The pair $(\mathcal{P}_n, (\mathcal{P}_n)^{\perp})$ is a complete and hereditary.

(1) Hovey's Criterion [2002]

If $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs in $\mathcal{A}$, and the class $\mathcal{W}$ is thick, then there exists a unique abelian model structure on $\mathcal{A}$ such that $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ are the classes of cofibrant objects, fibrant objects, and trivial objects, respectively.

Degreewise n-projective complexes

- Given a class $\mathcal{C}$ in R-Mod, a chain complex X is a *degreewise $\mathcal{C}$-complex* (denoted $X \in \mathrm{dw}(\mathcal{C})$) if $X_m \in \mathcal{C}$, $\forall\ m \in \mathbb{Z}$. Denote $\mathrm{ex}(\mathcal{C}) = \mathrm{dw}(\mathcal{C}) \cap \mathcal{E}$, where $\mathcal{E}$ is the class of *exact chain complexes*.
- **Example:** Every chain complex $X \in \mathrm{dw}(\mathcal{P}_n)$ is called *degreewise n-projective*.

(2) Proposition [Gillespie, 2008]

Let $(\mathcal{C}, \mathcal{F})$ be a cotorsion pair of modules. Then $(\mathrm{dw}(\mathcal{C}), (\mathrm{dw}(\mathcal{C}))^{\perp})$ and $(\mathrm{ex}(\mathcal{C}), (\mathrm{ex}(\mathcal{C}))^{\perp})$ are cotorsion pairs of complexes.

Filtrations

- Let $\mathcal{A}$ be either R-Mod or $\mathrm{Ch}(R)$, and $\mathcal{S}$ be a class of objects in $\mathcal{A}$. Given an object $X \in \mathcal{A}$, an *$\mathcal{S}$-filtration* of X indexed by an ordinal λ is a family $\{X^{\alpha}\}_{\alpha<\lambda}$ of subobjects of X such that:
  - (a) $X = \cup_{\alpha<\lambda} X_{\alpha}$.
  - (b) X^{α} is a subobject of $X^{\alpha'}$ whenever $\alpha \leq \alpha'$.
  - (c) $X^{\beta} = \cup_{\alpha<\beta} X^{\alpha}$ for any limit ordinal $\beta < \lambda$.
  - (d) For each $\alpha + 1 < \lambda$, X_0 and $X^{\alpha+1}/X^{\alpha}$ are isomorphic to an element in $\mathcal{S}$.
- Denote $\mathcal{S}^{\leq \aleph_0} := \{S \in \mathcal{S} : \mathrm{Card}(S) \leq \aleph_0\}$.

(3) Kaplansky's Theorem for $\mathcal{P}_n$ [Pérez, 2012]

Let R be a noetherian ring. Let $M \in \mathcal{P}_n$ and N be a countably generated submodule of M. Then there exists a $\mathcal{P}_n^{\leq\aleph_0}$-filtration of M, say $\{M_{\alpha}\}_{\alpha<\lambda}$ with $\lambda > 1$, such that $M_1 \in \mathcal{P}_n^{\leq\aleph_0}$ and $N \subseteq M_1$.

(4) Theorem [Pérez, 2012]

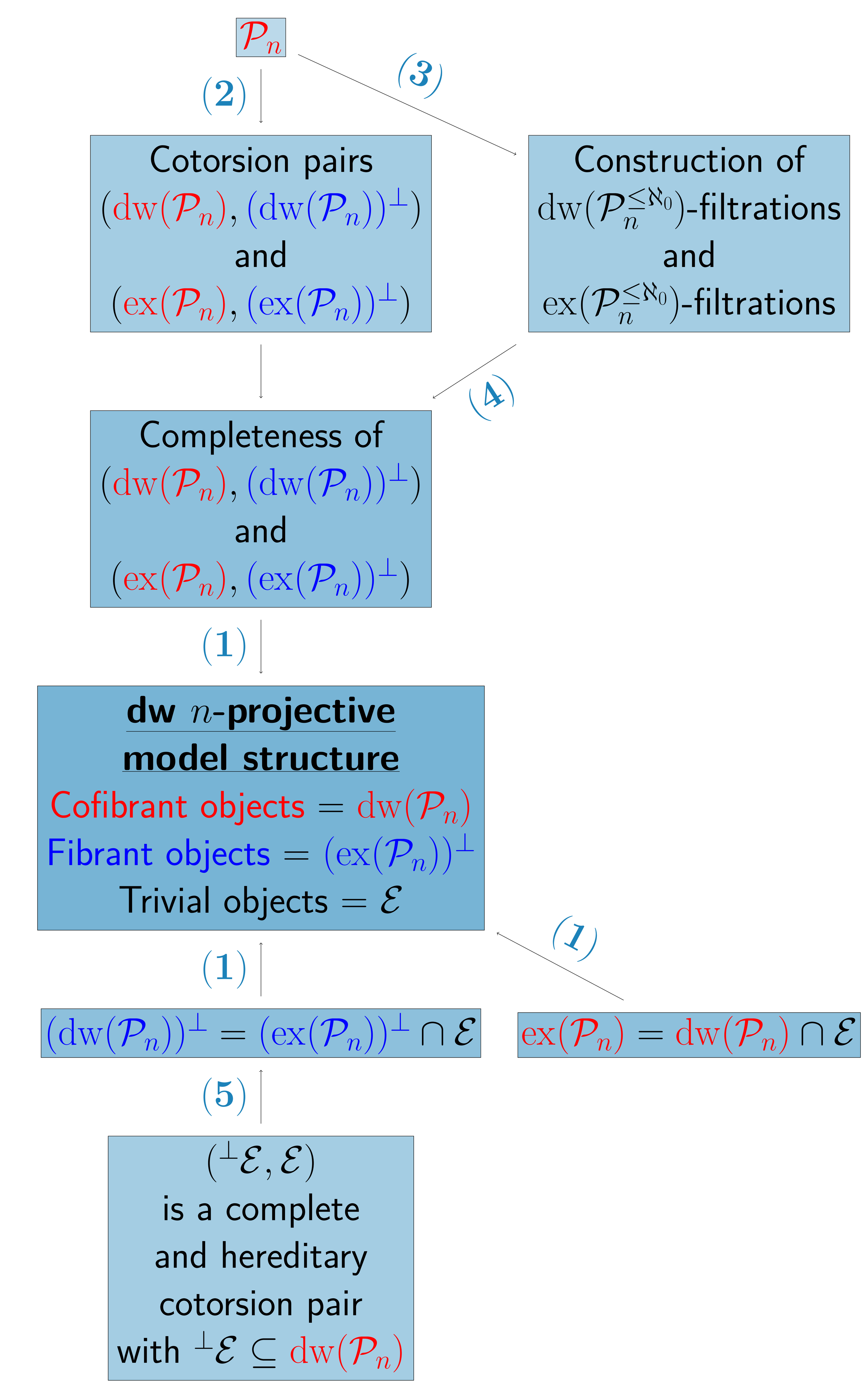
(a) Every $X \in \mathrm{dw}(\mathcal{P}_n)$ (resp. $X \in \mathrm{ex}(\mathcal{P}_n)$) has a $\mathrm{dw}(\mathcal{P}_n^{\leq\aleph_0})$-filtration (resp. an $\mathrm{ex}(\mathcal{P}_n^{\leq\aleph_0})$-filtration).

(b) $(\mathrm{dw}(\mathcal{P}_n), (\mathrm{dw}(\mathcal{P}_n))^{\perp})$ and $(\mathrm{ex}(\mathcal{P}_n), (\mathrm{ex}(\mathcal{P}_n))^{\perp})$ are complete cotorsion pairs.

Construction of the degreewise n-projective model structure

(5) Lemma [Rada et al, 2010]

If $(\mathcal{C}, \mathcal{F}')$ and $(\mathcal{U}, \mathcal{V})$ are cotorsion pairs of complexes such that $(\mathcal{U}, \mathcal{V})$ is complete and hereditary with $\mathcal{U} \subseteq \mathcal{C}$, then when $(\mathcal{C} \cap \mathcal{V})^{\perp} = \mathcal{F}$ we have $\mathcal{F}' = \mathcal{F} \cap \mathcal{V}$.



References

[1] D. Bravo, E. E. Enochs, A. C. Iacob, O. M. G. Jenda & J. Rada. *Cotorsion pairs in C(R-Mod)*. To appear in Rocky Mountain Journal of Mathematics.

[2] J. Gillespie. *Cotorsion Pairs and Degreewise Homological Model Structures*. Homology, Homotopy and Applications, Vol. 10, No. 1, pp. 283-304. (2008).

[3] M. Hovey. *Cotorsion pairs, model category structures and representation theory*. Mathematische Zeitschrift. Volume 241, Issue 3, pp. 553-592. (2002).

[4] M. Pérez. *Degreewise n-projective and n-flat model structures on chain complexes*. Preprint (2012).