The n-projective cotorsion pair of chain complexes

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Abstract

Let $\mathcal{P}_n(\mathbf{Ch}(R))$ denote the class of n-projective chain complexes, i.e. those complexes whose projective dimension is at most n. We show that the cotorsion pair $(\mathcal{P}_n(\mathbf{Ch}(R)), (\mathcal{P}_n(\mathbf{Ch}(R)))^{\perp})$ is cogenerated by a set (and so complete). Our proof consists in proving that every n-projective complex is filtered by the set of n-projective complexes with cardinality $\leq \kappa$, where κ is a infinite cardinal greater that $\mathrm{Card}(R)$. In the process, we show how to extend Enochs' zig-zag argument from the category of modules to the category of chain complexes.

1 Preliminaries

1.1 Chain complexes

Let R be any associative ring with identity. We use the following notation throughout these notes:

- ${}_R\mathbf{Mod}$ denotes the category of left R-modules and homomorphisms.
- $\mathbf{Ch}(R)$ denotes the category of chain complexes of left R-modules and chain maps.

Given a chain complex $X = (X_m)_{m\in\mathbb{Z}}$ with boundary maps $\partial^X_m : X_m \longrightarrow X_{m-1}$, the module

$$Z_m(X) := \mathrm{Ker}(\partial^X_m)$$

is called the m**-cycle** of X, and

$$B_m(X) := \mathrm{Im}(\partial^X_{m+1})$$

the m**-boundary** of X. A chain complex X is said to be **exact** if $Z_m(X) = B_m(X)$, for every $m \in \mathbb{Z}$.

A chain map $f : X \longrightarrow Y$ is a **monomorphism** (resp. **epimorphism**) if each $f_m : X_m \longrightarrow Y_m$ is an injective (resp. surjective) homomorphism, or equivalently if the sequence

$$0 \longrightarrow X \xrightarrow{f} Y \text{ (resp. } X \xrightarrow{f} Y \longrightarrow 0)$$

is exact.

Given a chain complex X, a complex X' is said to be a **subcomplex** of X if there exists a monomorphism $i : X' \longrightarrow X$. If X' is a subcomplex of X, we define the **quotient complex** X/X' as the complex whose components are given by

$$(X/X')_m = X_m/X'_m$$

and whose boundary maps $\partial_m^{X/X'} : X_m/X'_m \longrightarrow X_{m-1}/X'_{m-1}$ are given by

$$\partial_m^{X/X'} : x + X'_m \mapsto \partial_m^X(x) + X'_{m-1}.$$

Given a chain map $f : X \longrightarrow Y$, the **image complex** $\mathrm{Im}(f)$ is the chain complex given by

$$(\mathrm{Im}(f))_m := \mathrm{Im}(f_m) = f_m(X_m),$$

whose boundary maps $\partial_m^{\mathrm{Im}(f)} : f_m(X_m) \longrightarrow f_{m-1}(X_{m-1})$ are defined by

$$\partial_m^{\mathrm{Im}(f)} : x \mapsto \partial_m^Y \circ f_m(x) = f_{m-1} \circ \partial_m^X(x).$$

The **kernel complex** $\mathrm{Ker}(f)$ is the chain complex given by

$$(\mathrm{Ker}(f))_m := \mathrm{Ker}(f_m),$$

whose boundary maps $\partial_m^{\mathrm{Ker}(f)} : \mathrm{Ker}(f_m) \longrightarrow \mathrm{Ker}(f_{m-1})$ are defined by

$$\partial_m^{\mathrm{Ker}(f)} : x \mapsto \partial_m^X \circ i_m(x),$$

where i_m denotes the inclusion $\mathrm{Ker}(f_m) \longrightarrow X_m$.

1.2 Cotorsion pairs

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of chain complexes. The pair $(\mathcal{A}, \mathcal{B})$ is called a **cotorsion pair** if the following conditions are satisfied:

(1) $\mathcal{A} = {}^{\perp}\mathcal{B} := \{X \in \mathbf{Ch}(R) \ / \ \mathrm{Ext}^1(X, B) = 0 \text{ for every } B \in \mathcal{B}\}$.

(2) $\mathcal{B} = \mathcal{A}^{\perp} := \{X \in \mathbf{Ch}(R) \ / \ \mathrm{Ext}^1(A, X) = 0 \text{ for every } A \in \mathcal{A}\}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be **complete** if any of the two equivalent conditions holds:

(a) $(\mathcal{A}, \mathcal{B})$ has **enough projectives**: for every object X there exist objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and a short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow X \longrightarrow 0.$$

(b) $(\mathcal{A}, \mathcal{B})$ has **enough injectives**: for every object X there exist objects $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$, and a short exact sequence

$$0 \longrightarrow X \longrightarrow B' \longrightarrow A' \longrightarrow 0.$$

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be **cogenerated** by a set $\mathcal{S} \subseteq \mathcal{A}$ if $\mathcal{B} = \mathcal{S}^{\perp}$.

The previous definitions also hold for left R-modules.

There is a wide range of complete cotorsion pairs, thanks to the following result, known as the Eklof and Trlijaf Theorem:

Theorem 1.2.1. [2, Theorem 10] *Every cotorsion pair in R-Mod cogenerated by a set is complete.*

The previous theorem is also valid for chain complexes.

2 The cotorsion pair $(\mathcal{P}_n(\mathbf{Ch}(R)), (\mathcal{P}_n(\mathbf{Ch}(R)))^{\perp})$

2.1 n-projective modules

Every left R-module M has a projective resolution, i.e. an exact sequence

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where each P_i is a projective module. The **projective dimension** of M is defined as the integer

$$\mathrm{pd}(M) := \min\{n \geq 0 : \mathrm{Ext}^j(M, -) = 0,\ \forall j > n\},$$

provided such an integer exists. Otherwise, set $\mathrm{pd}(M) = \infty$. It is known that $\mathrm{pd}(M) \leq n$ if and only if there exists a finite projective resolution of length n, i.e. an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_i is projective for every $0 \leq i \leq n$. A module M is called n**-projective** if $\mathrm{pd}(M) \leq n$. We denote the class of n-projective modules by $\mathcal{P}_n({}_R\mathbf{Mod})$. The above comments and definition also hold for chain complexes, and we denote by $\mathcal{P}_n(\mathbf{Ch}(R))$ the class of n-projective complexes.

Given a left R-module M, the nth **disk complex** centred at M is the chain complex of the form

$$D^m(M) := \cdots \longrightarrow 0 \longrightarrow M \xrightarrow{=} M \longrightarrow 0 \longrightarrow \cdots$$

where M appears at the m-th and $m-1$-th entries.

Suppose we are given an n-projective module M with a projective resolution of length n as above. By Eilenberg's Trick, for each i there exists a free module F_i such that $P_i \oplus F_i \cong F_i$. Consider the disks complexes $D^{i+1}(F_i)$ where $0 \leq i \leq n-1$, and $D^n(F_n)$. Taking the direct sum of these disks and the given projective resolution of M, we get an exact sequence

$$0 \to P_n \oplus F_n \oplus F_{n-1} \to P_{n-1} \oplus F_{n-1} \oplus F_n \to \cdots \to P_0 \oplus F_0 \to M \to 0$$

which turns out to be a free resolution of M.

From now on, let κ be a fixed infinite cardinal such that $\kappa \geq \mathrm{Card}(R)$. We shall say that a set S is **small** if $\mathrm{Card}(S) \leq \kappa$.

The following result is due to Aldrich et al. (see [1, Proposition 4.1]). It is a tool used to prove that $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is a complete cotorsion pair.

Example 2.1.1. *The class $\mathcal{P}_n({}_R\mathbf{Mod})$ is the left half of a complete and hereditary cotorsion pair $(\mathcal{P}_n({}_R\mathbf{Mod}), (\mathcal{P}_n({}_R\mathbf{Mod}))^{\perp})$. Moreover, it is cogenerated by the set*

$$\mathcal{S} = \{L \in \mathcal{P}_n({}_R\mathbf{Mod}) : L \textit{ is small}\}.$$

The reader can check the details in [3, Theorem 7.4.6] *or in* [1, Theorem 4.2].

2.2 n-projective complexes

The following definition are due to J. Gillespie [4, Definitions 3.3]: Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair of left R-modules. Define $\widetilde{\mathcal{A}}$ as the class of all exact complexes X such that $Z_m(X) \in \mathcal{A}$ (resp. $Z_m(X) \in \mathcal{B}$) for every $m \in \mathbb{Z}$.

Proposition 2.2.1 (Gillespie, [4, Proposition 3.6])**.** *If $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in ${}_R\mathbf{Mod}$, then $(\widetilde{\mathcal{A}}, (\widetilde{\mathcal{A}})^{\perp})$ is a cotorsion pair in $\mathbf{Ch}(R)$.*

We know that $(\widetilde{\mathcal{P}_n({}_R\mathbf{Mod}}, (\widetilde{\mathcal{P}_n({}_R\mathbf{Mod})})^{\perp})$ is a complete cotorsion pair. So $(\widetilde{\mathcal{P}_n({}_R\mathbf{Mod})}, (\widetilde{\mathcal{P}_n({}_R\mathbf{Mod})})^{\perp})$ is a cotorsion pair in $\mathbf{Ch}(R)$. To show $\mathcal{P}_n(\mathbf{Ch}(R))$ is the left half of a complete cotorsion pair $(\mathcal{P}_n(\mathbf{Ch}(R)), (\mathcal{P}_n(\mathbf{Ch}(R)))^{\perp})$, we show that the classes $\widetilde{\mathcal{P}_n({}_R\mathbf{Mod})}$ and $\mathcal{P}_n(\mathbf{Ch}(R))$ coincide and that the former pair is complete.

We first study the case $n = 0$. In this case we get the projective cotorsion pair $(\mathcal{P}_0({}_R\mathbf{Mod}), {}_R\mathbf{Mod})$, which induces a cotorsion pair $(\mathcal{P}_0(\widetilde{{}_R\mathbf{Mod}}), (\mathcal{P}_0(\widetilde{{}_R\mathbf{Mod}}))^{\perp})$. The class $\mathcal{P}_0(\widetilde{{}_R\mathbf{Mod}})$ coincides with the class of projective chain complexes (See [5, Section 10.5]). It follows that $(\mathcal{P}_0(\mathbf{Ch}(R)), (\mathcal{P}_0(\mathbf{Ch}(\mathbf{R})))^{\perp})$ is a complete cotorsion pair.

Now we study the case when $n > 0$. Before proving the completeness of $(\widetilde{\mathcal{P}_n}, \mathrm{dg}\widetilde{\mathcal{P}_n^{\perp}})$, we need some lemmas. The following lemma follows using induction and the fact that the class of exact chain complexes is closed under cokernel of monomorphisms in $\mathcal{E}$, i.e that if $X \longrightarrow Y$ is a monomorphism in $\mathbf{Ch}(R)$ with X and Y exact, then $\mathrm{CoKer}(X \longrightarrow Y)$ is also exact.

Lemma 2.2.1. *Let*

$$0 \longrightarrow A^n \xrightarrow{f^n} A^{n-1} \longrightarrow \cdots \longrightarrow A^1 \xrightarrow{f^1} A^0 \xrightarrow{f^0} X \longrightarrow 0$$

be an exact sequence of chain complexes such that A^i is exact for every $0 \leq i \leq n$. Then X is also exact.

Lemma 2.2.2. *Consider a short exact sequence*

$$0 \longrightarrow Y \xrightarrow{f} Z \xrightarrow{g} X \longrightarrow 0$$

of chain complexes over R. Then the sequence

$$0 \longrightarrow Z_m(Y) \longrightarrow Z_m(Z) \longrightarrow Z_m(X) \longrightarrow 0$$

is exact if Y is an exact complex.

Proof. Let $Z_m(f) : Z_m(Y) \longrightarrow Z_m(Z)$ be the homomorphism induced by the universal property of kernels, given by $y \mapsto f_m(y)$ for every $y \in Z_m(Y)$. The homomorphism $Z_m(g) : Z_m(Y) \longrightarrow Z_m(X)$ is defined similarly. It is easy to check that $Z_m(f)$ is monic and that $\mathrm{Ker}(Z_m(g)) = \mathrm{Im}(Z_m(f))$. These facts do not depend on the exactness of Y. Let $x \in Z_m(X)$. There exists $z \in Z_m$ such that $x = g_m(z)$. We have $g_{m-1} \circ \partial_m^Z(z) = \partial_m^X \circ g_m(z) = 0$. Since the sequence $0 \longrightarrow Y_{m-1} \longrightarrow Z_{m-1} \longrightarrow X_{m-1} \longrightarrow 0$ is exact, there exists $y \in Y_{m-1}$ such that $\partial_m^Z(z) = f_{m-1}(y)$. Then $f_{m-2} \circ \partial_{m-1}^Y(y) = \partial_{m-1}^Z \circ f_{m-1}(y) = 0$ and so $\partial_{m-1}^Y(y) = 0$ since f_{m-2} is monic. By the exactness of Y, there exists $y' \in Y_m$ such that $y = \partial_m^Y(y')$. Hence $\partial_m^Z(z - f_m(y')) = 0$ and $g_m(z - f_m(y')) = x$. □

Using the previous lemma along with the induction principle, we obtain the following result.

Lemma 2.2.3. *Let*

$$0 \longrightarrow A^n \xrightarrow{f^n} A^{n-1} \longrightarrow \cdots \longrightarrow A^1 \xrightarrow{f^1} A^0 \longrightarrow 0$$

be an exact sequence of exact chain complexes. Then, for every $m \in \mathbb{Z}$, there exists an exact sequence of modules

$$0 \longrightarrow Z_m(A^n) \longrightarrow Z_m(A^{n-1}) \longrightarrow \cdots \longrightarrow Z_m(A^1) \longrightarrow Z_m(A^0) \longrightarrow 0.$$

Proposition 2.2.2. *A chain complex X is n-projective if, and only if, X is exact and each m-cycle is an n-projective module.*

Proof. Let X be an exact complex with n-projective cycles. Consider a partial projective resolution

$$0 \longrightarrow K \longrightarrow P^{n-1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow X \longrightarrow 0.$$

Note K is exact by Lemma 2.2.1. Notice also that $Z_m(P^i)$ is projective for every $0 \leq i \leq n-1$ and every $m \in \mathbb{Z}$. It follows that $Z_m(K)$ is projective since $Z_m(X)$ is n-projective. The converse follows similarly. □

Therefore, $\mathcal{P}_n(\widetilde{{}_R\mathbf{Mod}})$ is the class $\mathcal{P}_n(\mathbf{Ch}(R))$ of n-projective chain complexes.

2.3 Small n-projective filtrations

Given a chain complex $F \in \mathbf{Ch}(R)$, we shall say that F is a **free complex** if F is exact and each $Z_n(F)$ is a free left R-module. Note that every free complex is projective.

Lemma 2.3.1. *Every free complex can be decomposed into a direct sum of free disks. Conversely, any direct sum of free disks is free.*

Proof. Let F be a free complex. It is not hard to see that $F \cong \bigoplus_{m \in \mathbb{Z}} D^{m+1}(Z_m(F))$, where each $D^{m+1}(Z_m(F))$ is a free disk.

Now let $F = \bigoplus_{m \in \mathbb{Z}} D^{m+1}(F_m)$, where each F_m is a free module. It is clear that F is exact. Note that F has the form

$$\cdots \longrightarrow F_m \oplus F_{m+1} \xrightarrow{\partial^F_{m+1}} F_{m-1} \oplus F_m \xrightarrow{\partial^F_m} F_{m-2} \oplus F_{m-1} \longrightarrow \cdots$$

where each boundary map $\partial^F_m : F_{m-1} \oplus F_m \longrightarrow F_{m-2} \oplus F_{m-1}$ is given by $(x, y) \mapsto (0, x)$. We have that $Z_m(F) \cong F_m$ is a free module. □

Proposition 2.3.1 (Eilenberg Trick's in $\mathbf{Ch}(R)$)**.** *If P is a projective complex, then there exists a free complex F such that $P \oplus F \cong F$.*

Proof. Let P be a projective module. Then P is isomorphic to a direct sum of projective disks $P \cong \bigoplus_{m\in\mathbb{Z}} D^{m+1}(Z_m(P))$. By Eilenberg Trick's in R-Mod there exists a free module F_m such that $Z_m(P) \oplus F_m \cong F_m$. It follows

$$D^{m+1}(Z_m(P)) \oplus D^{m+1}(F_m) \cong D^{m+1}(F_m).$$

Setting $F = \bigoplus_{m\in\mathbb{Z}} D^{m+1}(F_m)$, which is a free complex by the previous lemma, we obtain

$$\begin{aligned} P \oplus F &\cong \left(\left(\bigoplus_{m\in\mathbb{Z}} D^{m+1}(Z_m(P))\right) \oplus \left(\bigoplus_{m\in\mathbb{Z}} D^{m+1}(F_m)\right)\right) \\ &\cong \bigoplus_{m\in\mathbb{Z}} \left(D^{m+1}(Z_m(P)) \oplus D^{m+1}(F_m)\right) \cong \bigoplus_{m\in\mathbb{Z}} D^{m+1}(F_m) = F. \end{aligned}$$

□

Lemma 2.3.2. *Every n-projective complex has a free resolution of length n.*

Proof. We only prove the case $n = 1$. The general case can be proven similarly. Let $X \in \mathcal{P}_1({}_R\mathbf{Mod})$ and let $0 \longrightarrow P^1 \longrightarrow P^0 \longrightarrow X \longrightarrow 0$ be a projective resolution of X of length 1. By Eilenberg's Trick in $\mathbf{Ch}(R)$, there exist free complexes F^0 and F^1 such that $P^0 \oplus F^0 \cong F^0$ and $P^1 \oplus F^1 \cong F^1$. Consider the short exact sequences

$$0 \longrightarrow F^1 \longrightarrow F^1 \longrightarrow 0 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F^0 \longrightarrow F^0 \longrightarrow 0 \longrightarrow 0.$$

Adding the sequences above, we get

$$\begin{gathered} 0 \longrightarrow P^1 \oplus F^1 \oplus F^0 \longrightarrow P^0 \oplus F^0 \oplus F^1 \longrightarrow X \longrightarrow 0 \\ \cong \\ 0 \longrightarrow F^1 \oplus F^0 \longrightarrow F^0 \oplus F^1 \longrightarrow X \longrightarrow 0. \end{gathered}$$

□

Lemma 2.3.3. *Given the following commutative diagram with exact rows and columns:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^n_m \oplus F^n_{m+1} & \xrightarrow{\partial^n_{m+1}} & F^n_{m-1} \oplus F^n_m & \xrightarrow{\partial^n_m} & F^n_{m-2} \oplus F^n_{m-1} & \longrightarrow & \cdots \\
 & & \downarrow f^n_{m+1} & & \downarrow f^n_m & & \downarrow f^n_{m-1} & & \\
\cdots & \longrightarrow & F^{n-1}_m \oplus F^{n-1}_{m+1} & \xrightarrow{\partial^{n-1}_{m+1}} & F^{n-1}_{m-1} \oplus F^{n-1}_m & \xrightarrow{\partial^{n-1}_m} & F^{n-1}_{m-2} \oplus F^{n-1}_{m-1} & \longrightarrow & \cdots \\
 & & \downarrow f^{n-1}_{m+1} & & \downarrow f^{n-1}_m & & \downarrow f^{n-1}_{m-1} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow f^2_{m+1} & & \downarrow f^2_m & & \downarrow f^2_{m-1} & & \\
\cdots & \longrightarrow & F^1_m \oplus F^1_{m+1} & \xrightarrow{\partial^1_{m+1}} & F^1_{m-1} \oplus F^1_m & \xrightarrow{\partial^1_m} & F^1_{m-2} \oplus F^1_{m-1} & \longrightarrow & \cdots \\
 & & \downarrow f^1_{m+1} & & \downarrow f^1_m & & \downarrow f^1_{m-1} & & \\
\cdots & \longrightarrow & F^0_m \oplus F^0_{m+1} & \xrightarrow{\partial^0_{m+1}} & F^0_{m-1} \oplus F^0_m & \xrightarrow{\partial^0_m} & F^0_{m-2} \oplus F^0_{m-1} & \longrightarrow & \cdots \\
 & & \downarrow f^0_{m+1} & & \downarrow f^0_m & & \downarrow f^0_{m-1} & & \\
\cdots & \longrightarrow & X_{m+1} & \xrightarrow{\partial^X_{m+1}} & X_m & \xrightarrow{\partial^X_m} & X_{m-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

For every $m \in \mathbb{Z}$ and every $i \in \{0, 1, \ldots, n\}$, one has $f^i_m(F^i_m) \subseteq F^{i-1}_m$. Moreover, the sequence

$$0 \longrightarrow F^n_{m-1} \oplus F^n_m \longrightarrow \cdots \longrightarrow F^1_{m-1} \oplus F^1_m \xrightarrow{f^1_m|_{F^1_m}} F^0_{m-1} \oplus F^0_m \xrightarrow{f^0_m|_{F^0_m}} X_m$$

is exact.

Proof. Let $(0, b) \in F^i_m$. We have $f^i_m(0, b) = f^i_m \circ \partial^i_{m+1}(b, 0) = \partial^{i-1}_{m+1} \circ f^i_{m+1}(b, 0) \in F^{i-1}_m$. It follows that $\mathrm{Im}(f^i_m|_{F^i_m}) \subseteq \mathrm{Ker}(f^{i-1}_m|_{F^{i-1}_m})$, for every $m \in \mathbb{Z}$. To prove the other inclusion, we start with $i = n$. Let $(0, b) \in F^n_m$ such that $f^n_m(0, b) = (0, 0)$. Then there exists $(\alpha, \beta) \in F^n_{m-1} \oplus F^n_m$ such that $(0, b) = f^n_m(\alpha, \beta)$, since the m-th column is exact. On the other hand,

$$f^n_{m-1}(0, \alpha) = f^n_{m-1} \circ \partial^n_m(\alpha, \beta) = \partial^{n-1}_m \circ f^n_m(\alpha, \beta) = \partial^{n-1}_m(0, b) = (0, 0).$$

Since f^n_{m-1} is injective, we have $(0, \alpha) = (0, 0)$ and so $(0, b) = f^n_m(0, \beta)$, i.e. $\mathrm{Im}(f^n_m|_{F^n_m})$ contains $\mathrm{Ker}(f^{n-1}_m|_{F^{n-1}_m})$, for every $m \in \mathbb{Z}$. Now we show that $\mathrm{Im}(f^{n-1}_m|_{F^{n-1}_m})$ contains $\mathrm{Ker}(f^{n-2}_m|_{F^{n-2}_m})$. Let $(0, b) \in \mathrm{Ker}(f^{n-2}_m)$. Since the central column is exact, there exists $(\alpha,\beta) \in F^{n-1}_{m-1} \oplus F^{n-1}_m$ such that $(0,b) = f^{n-1}_m(\alpha,\beta)$. On the other hand,

$$f^{n-1}_{m-1}(0,\alpha) = f^{n-1}_{m-1} \circ \partial^{n-1}_m(\alpha,\beta) = \partial^{n-2}_m \circ f^{n-1}_m(\alpha,\beta) = \partial^{n-2}_m(0,b) = (0,0).$$

Then $(0,\alpha) \in \mathrm{Ker}(f^{n-1}_{m-1}|_{F^{n-1}_{m-1}}) = \mathrm{Im}(f^n_{m-1}|_{F^n_{m-1}})$ and so there exists an element $(0,\gamma)$ in $F^n_{m-2} \oplus F^n_{m-1}$ such that $(0,\alpha) = f^n_{m-1}(0,\gamma)$. Since $(\alpha,\beta) - f^n_m(\gamma,0) \in F^{n-1}_{m-1} \oplus F^{n-1}_m$, we have

$$\begin{aligned}\partial^{n-1}_m((\alpha,\beta) - f^n_m(\gamma,0)) &= (0,\alpha) - \partial^{n-1}_m \circ f^n_m(\gamma,0)\\ &= (0,\alpha) - f^n_{m-1} \circ \partial^{n-2}_m(\gamma,0)\\ &= (0,\alpha) - f^n_{m-1}(0,\gamma) = (0,0),\end{aligned}$$

i.e. $(\alpha,\beta) - f^n_m(\gamma,0) \in \mathrm{Ker}(\partial^{n-1}_m) = F^{n-1}_m$. Also,

$$f^{n-1}_m((\alpha,\beta) - f^n_m(\gamma,0)) = f^{n-1}_m(\alpha,\beta) = (0,b).$$

Hence $\mathrm{Im}(f^{n-1}_m|_{F^{n-1}_m}) \supseteq \mathrm{Ker}(f^{n-2}_m|_{F^{n-2}_m})$. Repeating the same argument several times, we get the exact sequence

$$0 \longrightarrow F^n_{m-1} \oplus F^n_m \longrightarrow \cdots \longrightarrow F^1_{m-1} \oplus F^1_m \xrightarrow{f^1_m|_{F^1_m}} F^0_{m-1} \oplus F^0_m \xrightarrow{f^0_m|_{F^0_m}} X_m.$$

□

Given a chain complex $X = (X_m, \partial^X_m)_{m\in\mathbb{Z}}$, its cardinal number is defined as $\mathrm{Card}(X) = \sum_{m\in\mathbb{Z}} \mathrm{Card}(X_m)$. We shall say that X is **small** if $\mathrm{Card}(X) \leq \kappa$. We shall denote $x \in X$ whenever there exists $m \in \mathbb{Z}$ such that $x \in X_m$.

Lemma 2.3.4. *If X is an n-projective complex and $x \in X$, then there exists a small n-projective subcomplex $X' \subseteq X$ with $x \in X'$, such that X/X' is also n-projective.*

Proof. We only prove the case when $n = 1$. The general case follows similarly. Consider a free resolution of X of length 1 in $\mathbf{Ch}(R)$:

$$0 \longrightarrow L^1 \xrightarrow{f^1} L^0 \xrightarrow{f^0} X \longrightarrow 0.$$

The idea of the proof is to apply a generalization of the zig-zag argument to produce small free subcomplexes of L^0 and L^1, say $\overline{L}^0$ and $\overline{L}^1$, and a short exact sequence of the form

$$0 \longrightarrow \overline{L}^1 \xrightarrow{f^1|_{\overline{L}^1}} \overline{L}^0 \xrightarrow{f^0|_{\overline{L}^0}} X$$

where $X' = \mathrm{CoKer}(f^1|_{\overline{L}^1})$ is a subcomplex of X with $x \in X'$.

Since L^0 and L^1 are free complexes,

$$L^0 \cong \bigoplus_{i\in\mathbb{Z}} D^i(F^0_{i-1}) \quad \text{and} \quad L^1 \cong \bigoplus_{i\in\mathbb{Z}} D^i(F^1_{i-1}),$$

where F^0_i and F^1_i are free modules, for every $i \in \mathbb{Z}$. Let $\mathcal{B}^0_i$ and $\mathcal{B}^1_i$ be bases of F^0_i and F^1_i, respectively. Then $\mathcal{B}^0_{i-1} \sqcup \mathcal{B}^0_i$ and $\mathcal{B}^1_{i-1} \sqcup \mathcal{B}^1_i$ are bases of $F^0_{i-1} \oplus F^0_i$ and $F^1_{i-1} \oplus F^1_i$, respectively. Suppose $x \in X_m$. At the m-th level, we have the following exact sequence

$$0 \longrightarrow F^1_{m-1} \oplus F^1_m \xrightarrow{f^1_m} F^0_{m-1} \oplus F^0_m \xrightarrow{f^0_m} X_m \longrightarrow 0.$$

Consider the free resolution above as the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^1_m \oplus F^1_{m+1} & \xrightarrow{\partial^1_{m+1}} & F^1_{m-1} \oplus F^1_m & \xrightarrow{\partial^1_m} & F^1_{m-2} \oplus F^1_{m-1} & \longrightarrow & \cdots \\
 & & \downarrow f^1_{m+1} & & \downarrow f^1_m & & \downarrow f^1_{m-1} & & \\
\cdots & \longrightarrow & F^0_m \oplus F^0_{m+1} & \xrightarrow{\partial^0_{m+1}} & F^0_{m-1} \oplus F^0_m & \xrightarrow{\partial^0_m} & F^0_{m-2} \oplus F^0_{m-1} & \longrightarrow & \cdots \\
 & & \downarrow f^0_{m+1} & & \downarrow f^0_m & & \downarrow f^0_{m-1} & & \\
\cdots & \longrightarrow & X_{m+1} & \xrightarrow{\partial^X_{m+1}} & X_m & \xrightarrow{\partial^X_m} & X_{m-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let $W^0_{m-1} \sqcup W^0_m$ be a finite subset of $\mathcal{B}^0_{m-1} \sqcup \mathcal{B}^0_m$ (we mean $W^0_{m-1} \subseteq \mathcal{B}^0_{m-1}$ and $W^0_m \subseteq \mathcal{B}^0_m$) such that $x \in f^0_m(< W^0_{m-1} \sqcup W^0_m >)$. Now let $W^1_{m-1} \sqcup W^1_m \subseteq \mathcal{B}^1_{m-1} \sqcup \mathcal{B}^1_m$ be a small set such that

$$f^1_m \left(\langle W^1_{m-1} \sqcup W^1_m \rangle\right) \supseteq \operatorname{Ker}\left(f^0_m|_{\langle W^0_{m-1} \sqcup W^0_m \rangle}\right).$$

It is not necessarily true that $f^1_{m-1}(< W^1_{m-1} >) \supseteq \operatorname{Ker}(f^0_{m-1}|_{< W^0_{m-1} >})$. Since the sequence

$$0 \longrightarrow F^1_{m-1} \xrightarrow{f^1_{m-1}|_{F^1_{m-1}}} F^0_{m-1} \xrightarrow{f^0_{m-1}|_{F^0_{m-1}}} X_{m-1}$$

is exact, there exists a small set $\widetilde{W}^1_{m-1} \subseteq \mathcal{B}^1_{m-1}$ such that $f^1_{m-1}(< \widetilde{W}^1_{m-1} >)$ does contain $\operatorname{Ker}(f^0_{m-1}|_{< W^0_{m-1} >})$. Adding to $\widetilde{W}^1_{m-1}$ the elements of W^1_{m-1} which are not in $\widetilde{W}^1_{m-1}$, we may assume that $\widetilde{W}^1_{m-1} \supseteq W^1_{m-1}$. So we have

- $f^1_m(<\widetilde{W}^1_{m-1}\sqcup W^1_m>) \supseteq f^1_m(<W^1_{m-1}\sqcup W^1_m>) \supseteq \mathrm{Ker}(f^0_m|_{<W^0_{m-1}\sqcup W^0_m>})$.
- $f^1_{m-1}(<\widetilde{W}^1_{m-1}>) \supseteq \mathrm{Ker}(f^0_{m-1}|_{<W^0_{m-1}>})$.

Summarizing, we can choose a small set $W^1_{m-1}\sqcup W^1_m \subseteq \mathcal{B}^1_{m-1}\sqcup\mathcal{B}^1_m$ such that

- $f^1_m(<W^1_{m-1}\sqcup W^1_m>) \supseteq \mathrm{Ker}(f^0_m|_{<W^0_{m-1}\sqcup W^0_m>})$,
- $f^1_{m-1}(<W^1_{m-1}>) \supseteq \mathrm{Ker}(f^0_{m-1}|_{<W^0_{m-1}>})$.

Notice that $<W^j_{m-1}\sqcup W^j_m> \;=\; <W^j_{m-1}>\oplus<W^j_m>$, for $j=0,1$.

We go back to $F^0_{m-1}\oplus F^0_m$. Choose a small set $W^{0,(1)}_{m-1}\sqcup W^{0,(1)}_m \subseteq \mathcal{B}^1_{m-1}\sqcup\mathcal{B}^1_m$ containing $W^0_{m-1}\sqcup W^0_m$ (and so $W^{0,(1)}_{m-1}\supseteq W^0_{m-1}$ and $W^{0,(1)}_m\supseteq W^0_m$), such that

$$f^1_m\left(\langle W^1_{m-1}\rangle\oplus\langle W^1_m\rangle\right)\subseteq\left\langle W^{0,(1)}_{m-1}\right\rangle\oplus\left\langle W^{0,(1)}_m\right\rangle.$$

Note that $f^1_{m-1}(<W^1_{m-1}>)\subseteq<W^{0,(1)}_{m-1}>$.

Now we enlarge $W^1_{m-1}\sqcup W^1_m$, i.e. we choose a small set $W^{1,(1)}_{m-1}\sqcup W^{1,(1)}_m\subseteq\mathcal{B}^1_{m-1}\sqcup\mathcal{B}^1_m$ containing $W^1_{m-1}\sqcup W^1_m$ such that

- $f^1_m(<W^{1,(1)}_{m-1}>\oplus<W^{1,(1)}_m>)\supseteq\mathrm{Ker}(f^0_m|_{<W^{0,(1)}_{m-1}>\oplus<W^{0,(1)}_m>})$,
- $f^1_{m-1}(<W^{1,(1)}_{m-1}>)\supseteq\mathrm{Ker}(f^0_{m-1}|_{<W^{0,(1)}_{m-1}>})$.

Then enlarge $W^{0,(1)}_{m-1}\sqcup W^{0,(1)}_m$ to a small subset $W^{0,(2)}_{m-1}\sqcup W^{0,(2)}_m\subseteq\mathcal{B}^0_{m-1}\sqcup\mathcal{B}^0_m$ and so on. For $k=0,1$, set

- $B^{k,(0)}_{m-1}=\bigcup_{j=0}^\infty W^{k,(j)}_{m-1}$, where $W^{k,(0)}_{m-1}=W^k_{m-1}$ and $W^k_{m-1}\subseteq W^{k,(1)}_{m-1}\subseteq\cdots$,
- $B^{k,(0)}_m=\bigcup_{j=0}^\infty W^{k,(j)}_m$, where $W^{k,(0)}_m=W^k_m$ and $W^k_m\subseteq W^{k,(1)}_m\subseteq\cdots$.

Note that the sets above are linearly independent and small. By construction, we have

- $f^1_m(<B^{1,(0)}_{m-1}>\oplus<B^{1,(0)}_m>)\subseteq\;<B^{0,(0)}_{m-1}>\oplus<B^{0,(0)}_m>$, and
- $f^1_{m-1}(<B^{1,(0)}_{m-1}>)\subseteq\;<B^{0,(0)}_{m-1}>$.

Moreover, the following diagram commutes and has exact rows and columns:

$$
\begin{array}{ccccc}
 & & \left\langle B_m^{1,(0)} \right\rangle \oplus \langle 0 \rangle & & \left\langle B_m^{0,(0)} \right\rangle \oplus \langle 0 \rangle \\
 & & \downarrow \partial^1_{m+1} & & \downarrow \partial^0_{m+1} \\
0 & \longrightarrow & \left\langle B_{m-1}^{1,(0)} \right\rangle \oplus \left\langle B_m^{1,(0)} \right\rangle & \xrightarrow{f^1_m} & \left\langle B_{m-1}^{0,(0)} \right\rangle \oplus \left\langle B_m^{0,(0)} \right\rangle \\
 & & \downarrow \partial^1_m & & \downarrow \partial^0_m \\
0 & \longrightarrow & \langle 0 \rangle \oplus \left\langle B_{m-1}^{1,(0)} \right\rangle & \xrightarrow{f^1_{m-1}} & \langle 0 \rangle \oplus \left\langle B_{m-1}^{0,(0)} \right\rangle \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots
\end{array}
$$

where the morphisms appearing in it are their corresponding restrictions. At this point, the problem is that we do not know if the $m+1$-th row is exact. Actually, we do not even know if $f^1_{m+1}(< B_m^{1,(0)} >) \subseteq \; < B_m^{0,(0)} >$. In order to fix this problem, we are going to refine the sets of generators just obtained applying the zig-zag argument again, without destroying exactness in the other rows.

Choose a small set $Y^0_m \sqcup Y^0_{m+1} \subseteq \mathcal{B}^0_m \sqcup \mathcal{B}^0_{m+1}$ containing $B_m^{0,(0)}$ (and so $B_m^{0,(0)} \subseteq Y^0_m$) such that

$$f^1_{m+1}\left(\left\langle B_m^{1,(0)} \right\rangle\right) \subseteq \langle Y^0_m \rangle \oplus \langle Y^0_{m+1} \rangle .$$

Note that

$$f^1_m\left(\left\langle B_{m-1}^{1,(0)} \right\rangle \oplus \left\langle B_m^{1,(0)} \right\rangle\right) \subseteq \left\langle B_{m-1}^{0,(0)} \right\rangle \oplus \langle Y^0_m \rangle .$$

Now choose a small set $Y^1_m \sqcup Y^1_{m+1} \subseteq \mathcal{B}^1_m \sqcup \mathcal{B}^1_{m+1}$ containing $B_m^{1,(0)}$ (i.e. $B_m^{1,(0)} \subseteq Y^1_m$) such that

$$f^1_{m+1}\left(\langle Y^1_m \rangle \oplus \langle Y^1_{m+1} \rangle\right) \supseteq \operatorname{Ker}\left(f^0_{m+1}|_{\langle Y^0_m \rangle \oplus \langle Y^0_{m+1} \rangle}\right).$$

It is not necessarily true that

$$f^1_m\left(\left\langle B_{m-1}^{1,(0)} \right\rangle \oplus \langle Y^1_m \rangle\right) \supseteq \operatorname{Ker}\left(f^0_m|_{\left\langle B_{m-1}^{0,(0)} \right\rangle \oplus \langle Y^0_m \rangle}\right).$$

But there exists a small set $\widetilde{Y}^1_{m-1} \sqcup \widetilde{Y}^1_m \subseteq \mathcal{B}^1_{m-1} \sqcup \mathcal{B}^1_m$ containing $B_{m-1}^{1,(0)} \sqcup B_m^{1,(0)}$

such that

$$f_m^1\left(\left\langle\widetilde{Y}_{m-1}^1\right\rangle\oplus\left\langle\widetilde{Y}_m^1\right\rangle\right)\supseteq\mathrm{Ker}\left(f_m^0|_{\left\langle B_{m-1}^{0,(0)}\right\rangle\oplus\langle Y_m^0\rangle}\right).$$

We may assume that $\widetilde{Y}_m^1\supseteq Y_m^1$. So

$$f_{m+1}^1\left(\left\langle\widetilde{Y}_m^1\right\rangle\oplus\left\langle Y_{m+1}^1\right\rangle\right)\supseteq\mathrm{Ker}\left(f_{m+1}^0|_{\langle Y_m^0\rangle\oplus\left\langle Y_{m+1}^0\right\rangle}\right).$$

Note that

$$f_{m-1}^1\left(\left\langle\widetilde{Y}_{m-1}^1\right\rangle\right)\supseteq\mathrm{Ker}\left(f_{m-1}^0|_{\left\langle B_{m-1}^{0,(0)}\right\rangle}\right).$$

Summarizing, we may choose small sets $Y_{m-1}^1\subseteq\mathcal{B}_{m-1}^1$, $Y_m^1\subseteq\mathcal{B}_m^1$, $Y_{m+1}^1\subseteq\mathcal{B}_{m+1}^1$, containing $B_{m-1}^{1,(0)}$, $B_m^{1,(0)}$ and $B_{m+1}^{1,(0)}$, respectively, such that

- $f_{m+1}^1(<Y_m^1>\oplus<Y_{m+1}^1>)\supseteq\mathrm{Ker}(f_{m+1}^0|_{<Y_m^0>\oplus<Y_{m+1}^0>})$,
- $f_m^1(<Y_{m-1}^1>\oplus<Y_m^1>)\supseteq\mathrm{Ker}(f_m^0|_{<B_{m-1}^{0,(0)}>\oplus<Y_m^0>})$, and
- $f_{m-1}^1(<Y_{m-1}^1>)\supseteq\mathrm{Ker}(f_{m-1}^0|_{<B_{m-1}^{0,(0)}>})$.

Now choose a small set $Y_m^{0,(1)}\sqcup Y_{m+1}^{0,(1)}\subseteq\mathcal{B}_m^0\sqcup\mathcal{B}_{m+1}^0$ containing $Y_m^0\sqcup Y_{m+1}^0$ such that

$$f_{m+1}^1\left(\left\langle Y_m^1\right\rangle\oplus\left\langle Y_{m+1}^1\right\rangle\right)\subseteq\left\langle Y_m^{0,(1)}\right\rangle\oplus\left\langle Y_{m+1}^{0,(1)}\right\rangle.$$

On the other hand, choose a small set $B_{m-1}^{0,(0)}\sqcup Y_m^{0,(1)}\subseteq\widetilde{Y}_{m-1}^{0,(1)}\sqcup\widetilde{Y}_m^{0,(1)}\subseteq\mathcal{B}_{m-1}^0\sqcup\mathcal{B}_m^0$ such that

$$f_m^1\left(\left\langle Y_{m-1}^1\right\rangle\oplus\left\langle Y_m^1\right\rangle\right)\subseteq\left\langle\widetilde{Y}_{m-1}^{0,(1)}\right\rangle\oplus\left\langle\widetilde{Y}_m^{0,(1)}\right\rangle.$$

Note that $f_{m-1}^1(Y_{m-1}^1)\subseteq<\widetilde{Y}_{m-1}^{0,(1)}>$ and that we may choose $\widetilde{Y}_m^{0,(1)}$ containing $Y_m^{0,(1)}$. Summarizing, there exist small sets $Y_{m-1}^{0,(1)}\subseteq\mathcal{B}_{m-1}^0$, $Y_m^{0,(1)}\subseteq\mathcal{B}_m^0$ and $Y_{m+1}^{0,(1)}\subseteq\mathcal{B}_{m+1}^0$ containing $B_{m-1}^{0,(0)}$, Y_m^0 and Y_{m+1}^0, respectively, such that

- $f_{m+1}^1(<Y_m^1>\oplus<Y_{m+1}^1>)\subseteq<Y_m^{0,(1)}>\oplus<Y_{m+1}^{0,(1)}>$,
- $f_m^1(<Y_{m-1}^1>\oplus<Y_m^1>)\subseteq<Y_{m-1}^{0,(1)}>\oplus<Y_m^{0,(1)}>$, and
- $f_{m-1}^1(<Y_{m-1}^1>)\subseteq<Y_{m-1}^{0,(1)}>$.

Now choose small sets $Y_{m-1}^{1,(1)}\subseteq\mathcal{B}_{m-1}^1$, $Y_m^{1,(1)}\subseteq\mathcal{B}_m^{1,(1)}$ and $Y_{m+1}^{1,(1)}\subseteq\mathcal{B}_{m+1}^1$ containing Y_{m-1}^1, Y_m^1 and Y_{m+1}^1, respectively, such that

- $f_{m+1}^1(<Y_m^{1,(1)}>\oplus<Y_{m+1}^{1,(1)}>)\supseteq\mathrm{Ker}(f_{m+1}^0|_{<Y_m^{0,(1)}>\oplus<Y_{m+1}^{0,(1)}>})$, and
- $f_m^1(<Y_{m-1}^{1,(1)}>\oplus<Y_m^{1,(1)}>)\supseteq\mathrm{Ker}(f_m^0|_{<Y_{m-1}^{0,(1)}>\oplus<Y_m^{0,(1)}>})$.

It is not necessarily true that

$$f^1_{m-1}\left(Y^{1,(1)}_{m-1}\right) \supseteq \operatorname{Ker}\left(f^0_{m-1}|_{\left\langle Y^{0,(1)}_{m-1}\right\rangle}\right).$$

But using similar arguments as above, we can enlarge $Y^{1,(1)}_{m-1}$ in such a way that the previous inclusion is satisfied.

Continue using the zig-zag procedure to enlarge $Y^{0,(1)}_{m-1}$, $Y^{0,(1)}_{m}$, $Y^{0,(1)}_{m+1}$ and so on. Then set

- $B^{0,(1)}_{m-1} = Y^{0,(1)}_{m-1} \cup Y^{0,(2)}_{m-1} \cup \cdots,$
- $B^{1,(1)}_{m-1} = Y^{1}_{m-1} \cup Y^{1,(1)}_{m-1} \cup \cdots,$
- $B^{0,(1)}_{m} = Y^{0}_{m} \cup Y^{0,(1)}_{m} \cup \cdots,$
- $B^{1,(1)}_{m} = Y^{1}_{m} \cup Y^{1,(1)}_{m} \cup \cdots,$
- $B^{0,(1)}_{m+1} = Y^{0}_{m+1} \cup Y^{0,(1)}_{m+1} \cup \cdots,$
- $B^{1,(1)}_{m+1} = Y^{1}_{m+1} \cup Y^{1,(1)}_{m+1} \cup \cdots.$

Hence we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
 & & \left\langle B^{1,(1)}_{m+1}\right\rangle \oplus \langle 0\rangle & & \left\langle B^{0,(1)}_{m+1}\right\rangle \oplus \langle 0\rangle \\
 & & \Big\downarrow \partial^1_{m+2} & & \Big\downarrow \partial^0_{m+2} \\
0 & \longrightarrow & \left\langle B^{1,(1)}_{m}\right\rangle \oplus \left\langle B^{1,(1)}_{m+1}\right\rangle & \xrightarrow{f^1_{m+1}} & \left\langle B^{0,(1)}_{m}\right\rangle \oplus \left\langle B^{0,(1)}_{m+1}\right\rangle \\
 & & \Big\downarrow \partial^1_{m+1} & & \Big\downarrow \partial^0_{m+1} \\
0 & \longrightarrow & \left\langle B^{1,(1)}_{m-1}\right\rangle \oplus \left\langle B^{1,(1)}_{m}\right\rangle & \xrightarrow{f^1_{m}} & \left\langle B^{0,(1)}_{m-1}\right\rangle \oplus \left\langle B^{0,(1)}_{m}\right\rangle \\
 & & \Big\downarrow \partial^1_{m} & & \Big\downarrow \partial^0_{m} \\
0 & \longrightarrow & \langle 0\rangle \oplus \left\langle B^{1,(1)}_{m-1}\right\rangle & \xrightarrow{f^1_{m-1}} & \langle 0\rangle \oplus \left\langle B^{0,(1)}_{m-1}\right\rangle \\
 & & \Big\downarrow & & \Big\downarrow \\
 & & 0 & & 0 \\
 & & \Big\downarrow & & \Big\downarrow \\
 & & \vdots & & \vdots
\end{array}$$

As in the previous iteration, we do not know if

$$f^1_{m+2}\left(\left\langle B^{1,(1)}_{m+1}\right\rangle \oplus \langle 0\rangle\right) \subseteq \left\langle B^{0,(1)}_{m+1}\right\rangle \oplus \langle 0\rangle\,.$$

Then repeat the same process and so on. In the i-th iteration, we get, for $j = 0, 1$, small sets $B^{j,(k)}_{m+k}$ such that:

- $B^{j,(i)}_{m+i} \subseteq \mathcal{B}^j_{m+i}$,
- $B^{j,(i-1)}_{m+i-1} \subseteq B^{j,(i)}_{m+i-1} \subseteq \mathcal{B}^j_{m+i-1}$,
- $B^{j,(i-2)}_{m+i-2} \subseteq B^{j,(i-1)}_{m+i-2} \subseteq B^{j,(i)}_{m+i-2} \subseteq \mathcal{B}^j_{m+i-2}$,

  $\vdots$
- $B^{j,(0)}_{m} \subseteq \cdots \subseteq B^{j,(i-1)}_{m} \subseteq B^{j,(i)}_{m} \subseteq \mathcal{B}^j_{m}$,
- $B^{j,(0)}_{m-1} \subseteq \cdots \subseteq B^{j,(i-1)}_{m-1} \subseteq B^{j,(i)}_{m-1} \subseteq \mathcal{B}^j_{m-1}$.

and the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
 & & \left\langle B^{1,(i)}_{m+i}\right\rangle \oplus \langle 0\rangle & & \left\langle B^{0,(i)}_{m+i}\right\rangle \oplus \langle 0\rangle \\
 & & \downarrow \partial^1_{m+i+1} & & \downarrow \partial^0_{m+i+1} \\
0 & \longrightarrow & \left\langle B^{1,(i)}_{m+i-1}\right\rangle \oplus \left\langle B^{1,(i)}_{m+i}\right\rangle & \xrightarrow{f^1_{m+i}} & \left\langle B^{0,(i)}_{m+i-1}\right\rangle \oplus \left\langle B^{0,(i)}_{m+i}\right\rangle \\
 & & \downarrow \partial^1_{m+i} & & \downarrow \partial^0_{m+i} \\
 & & \vdots & & \vdots \\
 & & \downarrow \partial^1_{m+1} & & \downarrow \partial^0_{m+1} \\
0 & \longrightarrow & \left\langle B^{1,(i)}_{m-1}\right\rangle \oplus \left\langle B^{1,(i)}_{m}\right\rangle & \xrightarrow{f^1_{m}} & \left\langle B^{0,(i)}_{m-1}\right\rangle \oplus \left\langle B^{0,(i)}_{m}\right\rangle \\
 & & \downarrow \partial^1_{m} & & \downarrow \partial^0_{m} \\
0 & \longrightarrow & \langle 0\rangle \oplus \left\langle B^{1,(i)}_{m-1}\right\rangle & \xrightarrow{f^1_{m-1}} & \langle 0\rangle \oplus \left\langle B^{0,(i)}_{m-1}\right\rangle \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots
\end{array}$$

Finally, for $j = 0, 1$, set:

- $B^j_{m-1} = B^{j,(0)}_{m-1} \cup B^{j,(1)}_{m-1} \cup \cdots,$
- $B^j_{m} = B^{j,(0)}_{m} \cup B^{j,(1)}_{m} \cup \cdots,$
- $B^j_{m+1} = B^{j,(1)}_{m+1} \cup B^{j,(2)}_{m+1} \cup \cdots,$

  $\vdots$
- $B^j_{m+i} = B^{j,(i)}_{m+i} \cup B^{j,(i+1)}_{m+i} \cup \cdots,$

  $\vdots$

All of these sets are small. We have the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \left\langle B^1_{m+i} \right\rangle \oplus \left\langle B^1_{m+i+1} \right\rangle & \xrightarrow{f^1_{m+i+1}} & \left\langle B^0_{m+i} \right\rangle \oplus \left\langle B^0_{m+i+1} \right\rangle & \xrightarrow{f^0_{m+i+1}} & X_{m+i+1} \\
 & & \downarrow \partial^1_{m+i+1} & & \downarrow \partial^0_{m+i+1} & & \downarrow \partial^X_{m+i+1} \\
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow \partial^1_{m+1} & & \downarrow \partial^0_{m+1} & & \downarrow \partial^X_{m+1} \\
0 & \longrightarrow & \left\langle B^1_{m-1} \right\rangle \oplus \left\langle B^1_{m} \right\rangle & \xrightarrow{f^1_{m}} & \left\langle B^0_{m-1} \right\rangle \oplus \left\langle B^0_{m} \right\rangle & \xrightarrow{f^0_{m}} & X_{m} \\
 & & \downarrow \partial^1_{m} & & \downarrow \partial^0_{m} & & \downarrow \partial^X_{m} \\
0 & \longrightarrow & \langle 0 \rangle \oplus \left\langle B^1_{m-1} \right\rangle & \xrightarrow{f^1_{m-1}} & \langle 0 \rangle \oplus \left\langle B^0_{m-1} \right\rangle & \xrightarrow{f^0_{m-1}} & X_{m-1} \\
 & & \downarrow \partial^1_{m-1} & & \downarrow \partial^0_{m-1} & & \downarrow \partial^X_{m-1} \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & X_{m-2} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}
$$

We have obtained the following exact sequence in $\mathbf{Ch}(R)$:

$$
0 \longrightarrow \bigoplus_{i \geq m} D^i\left(\left\langle B^1_{i-1} \right\rangle\right) \xrightarrow{f^1|_{\bigoplus_{i \geq m} D^i\left(\left\langle B^1_{i-1} \right\rangle\right)}} \bigoplus_{i \geq m} D^i\left(\left\langle B^0_{i-1} \right\rangle\right) \longrightarrow X' \xrightarrow{f^0|_{\bigoplus_{i \geq m} D^i\left(\left\langle B^0_{i-1} \right\rangle\right)}} 0
$$

where $X' = \mathrm{CoKer}(f^1|_{\bigoplus_{i\geq m} D^i(<B^1_{i-1}>)})$. Note that $x \in X'$, that X' is a subcomplex of X, and that $\bigoplus_{i\geq m} D^i(< B^1_{i-1} >)$ and $\bigoplus_{i\geq m} D^i(< B^0_{i-1} >)$ are small subcomplexes of L^0 and L^1, respectively. Since $f^0|_{\bigoplus_{i\geq m} D^i(<B^0_{i-1}>)}$ is surjective, we have that X' is also small. The exact sequence above is a projective resolution of X' of length 1, hence X' is 1-projective.

Now consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i\geq m} D^i\left(\langle B^1_{i-1}\rangle\right) & \longrightarrow & \bigoplus_{i\geq m} D^i\left(\langle B^0_{i-1}\rangle\right) & \longrightarrow & X' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^1 & \xrightarrow{f^1} & L^0 & \xrightarrow{f^0} & X & \longrightarrow & 0
\end{array}$$

Taking the quotient of the resolution of X by the resolution of X', we get a free resolution of X/X', and so X/X' is also 1-projective. □

Given a chain complex $X \in \mathbf{Ch}(R)$ and an ordinal number λ, a **filtration of** X **indexed by** λ is a family $(X_\alpha)_{\alpha\leq\lambda}$ of subcomplexes of X such that

(a) $X_\lambda = X$,

(b) $X_0 = 0$,

(c) X_α is a subcomplex of $X_{\alpha'}$ whenever $\alpha \leq \alpha'$, and

(d) $X_\beta = \bigcup_{\alpha<\beta} X_\alpha$ whenever β is a limit ordinal.

By the **union of chain complexes** $X_\beta = \bigcup_{\alpha<\beta} X_\alpha$ we mean the chain complex whose objects are given by

$$X_{\beta,n} = \bigcup_{\alpha<\beta} X_{\alpha,n} = \{(\alpha, x) : \alpha < \beta \text{ and } x \in X_{\alpha,n}\},$$

and whose morphisms are given by

$$\begin{aligned} \partial^{X_\beta}_n : X_{\beta,n} &\longrightarrow X_{\beta,n-1} \\ (\alpha, x) &\longmapsto \left(\alpha, \partial^{X_\alpha}_n(x)\right). \end{aligned}$$

If $\mathcal{S}$ is some class of complexes in $\mathbf{Ch}(R)$, we shall say that a filtration $(X_\alpha)_{\alpha\leq\lambda}$ of X is an $\mathcal{S}$**-filtration** if for each $\alpha + 1 < \lambda$ we have that X_0 and $X_{\alpha+1}/X_\alpha$ are isomorphic to elements of $\mathcal{S}$.

Lemma 2.3.5 (Eklof)**.** *Let X and Y be chain complexes and let $(X_\alpha)_{\alpha\leq\lambda}$ be a ${}^\perp\{Y\}$-filtration of X. Then* $\mathrm{Ext}^1(X, Y) = 0$.

The proof of the previous result is given in [3, Theorem 7.3.4] in the category R-Mod carries over directly to the category $\mathbf{Ch}(R)$.

Theorem 2.3.1. *The cotorsion pair* $(\widetilde{\mathcal{P}_n}, \widetilde{\mathcal{P}_n}^{\perp})$ *is complete.*

Proof. Applying Lemma 2.3.4, we have that any complex $X \in \widetilde{\mathcal{P}_n}$ can be written as a union $X = \bigcup_{\alpha<\lambda} X_\alpha$, where $(X_\alpha)_{\alpha\leq\lambda}$ is a $\widetilde{\mathcal{P}_n}$-filtration, and X_0 and $X_{\alpha+1}/X_\alpha$ are small complexes whenever $\alpha + 1 < \lambda$. We construct such a filtration by using transfinite induction. Set $X_0 = 0$. Now choose any $x \neq 0$ in X and let X_1 be the complex given by Lemma 2.3.4. We have that $X_0, X_1 \in \widetilde{\mathcal{P}_n}$ and they are small. Also, $X/X_1 \in \widetilde{\mathcal{P}_n}$. If $X_1 \subsetneq X$ then choose $x' + X_1 \neq 0 + X_1$ in X/X_1. Using Lemma 2.3.4 again, we can construct a chain complex $X_2/X_1 \subseteq X/X_1$ such that $x' + X_1 \in X_2/X_1$, X_2/X_1 is small and $X_2/X_1 \in \widetilde{\mathcal{P}_n}$. Now suppose that β is an ordinal and that for any $\alpha < \beta$ one has constructed chain complexes $X_\alpha \subseteq X$ such that:

(a) $X_\alpha \subseteq X_{\alpha'}$ whenever $\alpha \leq \alpha'$,

(b) $X_{\alpha+1}/X_\alpha \in \widetilde{\mathcal{P}_n}$ and $X_{\alpha+1}/X_\alpha$ is small whenever $\alpha + 1 < \beta$,

(c) $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$ for every limit ordinal $\gamma < \beta$.

If β is a limit ordinal, set $X_\beta = \bigcup_{\alpha<\beta} X_\alpha$. Otherwise there exists an ordinal $\alpha < \beta$ such that $\beta = \alpha+1$. Then construct $X_{\alpha+1}$ from X_α as we just constructed X_2 from X_1. By transfinite induction, we have a $\widetilde{\mathcal{P}_n}$-filtration $(X_\alpha : \alpha < \lambda)$ of X such that $X_{\alpha+1}/X_\alpha$ is small whenever $\alpha + 1 < \lambda$. Now consider the set

$$\mathcal{S} = \left\{ L \in \widetilde{\mathcal{P}_n} \ / \ L \text{ is small complex} \right\}.$$

Note that $\widetilde{\mathcal{P}_n}^{\perp} \subseteq \mathcal{S}^{\perp}$ since $\mathcal{S} \subseteq \widetilde{\mathcal{P}_n}$. Now let $Y \in \mathcal{S}^{\perp}$ and $X \in \widetilde{\mathcal{P}_n}$. Write $X = \bigcup_{\alpha<\lambda} X_\alpha$ with $(X_\alpha)_{\alpha<\lambda}$ as above. Then $X_0, X_{\alpha+1}/X_\alpha \in \mathcal{S}$. Hence $\mathrm{Ext}^1(X_0, Y) = 0$ and $\mathrm{Ext}^1(X_{\alpha+1}/X_\alpha, Y) = 0$, i.e. $(X_\alpha)_{\alpha\leq\lambda}$ is a ${}^{\perp}\{Y\}$-filtration of X. By the Eklof Lemma, we have $\mathrm{Ext}^1(X, Y) = 0$ and so $\mathcal{S}^{\perp} \subseteq \widetilde{\mathcal{P}_n}^{\perp}$. We have $\widetilde{\mathcal{P}_n}^{\perp} = \mathcal{S}^{\perp}$. Therefore, by the Eklof and Trlifaj Theorem we conclude that $(\widetilde{\mathcal{P}_n}, \widetilde{\mathcal{P}_n}^{\perp})$ is a complete cotorsion pair. □

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