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THE RIEMANN-ROCH THEOREM

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Contents

Introduction

The Riemann-Roch theorem is an important result in complex geometry, for the computation of the dimension of the space of meromorphic functions with prescribed zeroes and allowed poles. Roughly speaking, it links the complex analysis of a compact Riemann surface with its topological genus. Initially proved as Riemann's inequality by Bernhard Riemann in 1857,

Theorem 1. Let X be a Riemann surface of genus g, then

$$\dim(L(D)) \geq \deg(D) + 1 - g,$$

where $L(D)$ is the space of meromorphic functions with poles bounded by a divisor D.

the theorem reached its definitive form thanks to the work of Gustav Roch, one of Riemann's students, in 1865 providing the error term:

Theorem 2. Let X be a Riemann surface of genus g. Then for any divisor D and any canonical divisor K, we have

$$\dim(L(D)) - \dim(L(K - D)) = \deg(D) + 1 - g.$$

Originally developed in complex function theory, Riemann and Roch proved the theorem in a purely analytic context. The legitimacy of the proof was called into question by Weierstrass, who found a counterexample to a main tool in the proof that Riemann called Dirichlet's principle. In spite of this challenge, the theorems ideas were too useful to do without.

The Riemann-Roch Theorem was proved for algebraic curves by F. K. Schmidt in 1929. It was later generalized to higher-dimensional varieties and beyond, using the appropriate notion of divisor, or line bundle. The general formulation of the theorem depends on splitting it into two parts. One, which would now be called Serre Duality, interprets the $L(K - D)$ term as a dimension of a first sheaf cohomology group; with $L(D)$ the dimension of a zeroth cohomology group, or space of sections, the left-hand side of the theorem becomes an Euler characteristic, and the right-hand side a computation of it as a degree corrected according to the topology of the Riemann surface.

An n-dimensional generalization, the Hirzebruch-Riemann-Roch Theorem, was found and proved by Friedrich Hirzebruch, as an application of characteristic classes in algebraic topology.

At about the same time Jean-Pierre Serre was giving the general form of Serre Duality. Alexander Grothendieck proved a far-reaching generalization in 1957, now known as the Grothendieck-Riemann-Roch Theorem. His work reinterprets Riemann-Roch not as a theorem about a variety, but about a morphism between two varieties.

These notes are divided into two chapters. In the first chapter we recall some notions of complex geometry. Then we give the concept of an algebraic curve and study some of its properties, such as the Laurent Series Approximation Theorem. In the last section of this chapter, we study the field of meromorphic functions on an algebraic curve X, $\mathcal{M}(X)$, as a field extension of certain subfield. Such a subfield is the field of rational expressions in a nonconstant meromorphic function on X. The degree of this extension shall be an important tool to prove the Riemann-Roch Theorem.

In the second chapter we prove the Riemann-Roch Theorem for algebraic curves, by studying the spaces $L(D)$ and $L^{(1)}(D)$ of meromorphic functions and meromorphic 1-forms, respectively, with poles bounded by a divisor D. We shall prove a result called the Serre Duality Theorem, which is the most important tool to prove the Riemann-Roch Theorem. Finally, we shall comment some of the generalizations of this theorem, such as the Riemann-Roch Theorem for holomorphic line bundles, the Hirzebruch-Riemann-Roch Theorem, and the Grothendick-Riemann-Roch Theorem.

Chapter 1

Algebraic curves and some properties of its function field

This chapter is devoted to the study of several properties of a certain type of Riemann surfaces called algebraic curves. In the first section we shall recall some notions of complex geometry, such as meromorphic functions, divisors, and meromorphic 1-forms. We shall also give the notion of an algebraic curve. These structures have very interesting properties. One of them is known as the Laurent Series Approximation Theorem. Roughly speaking, this theorem allows us to construct a meromorphic function on X from a finite number of points in X and tails of Laurent series such that at each of these points the Laurent series of the function starts with one of these tails. Another property of the algebraic curves is that one can always find a nonconstant meromorphic function on it. So given a nonconstant meromorphic function f on al algebraic curve X, one can consider the field $\mathbb{C}(f)$ of rational expressions of f with coefficients in $\mathbb{C}$, which is a subfield of the field $\mathcal{M}(X)$ of meromorphic functions on X. The Laurent Series Approximation Theorem shall help us to compute the degree of the extension $\mathcal{M}(X)/\mathbb{C}(f)$.

1.1 Preliminaries

Recall that a function $f : X \longrightarrow \mathbb{C}$ is **meromorphic** at $p \in X$ if it is either holomorphic, has a removable singularity, or has a pole at p.

Given a function $D : X \longrightarrow \mathbb{Z}$ on a Riemann surface X, the set of all p such that $D(p) \neq 0$ is called the **support** of D. A **divisor on** X is a function $D : X \longrightarrow \mathbb{Z}$ whose support is a discrete subset of X. We shall denote a divisor D by using the following summation notation:

$$D = \sum_{p \in X} D(p) \cdot p.$$

The divisor of a meromorphic function $f : X \longrightarrow \mathbb{C}$, denoted by (f), is the divisor defined by the order function:

$$(f) = \sum_{p \in X} \text{ord}_p(f) \cdot p.$$

The **divisor of poles of** f, denoted by $(f)_\infty$, is the divisor

$$(f)_\infty = \sum_{p \text{ with } \text{ord}_p(f)<0} (-\text{ord}_p(f)) \cdot p.$$

The **space of meromorphic functions with poles bounded by** D, denoted $L(D)$, is the set of meromorphic functions

$$L(D) = \{f \in \mathcal{M}(X) \ / \ (f) \geq -D\},$$

where $(f) \geq -D$ means $(f)(p) \geq -D(p)$ for every $p \in X$.

Proposition 1.1.1. Let X be a compact Riemann surface, and let D be a divisor on X. Then the space $L(D)$ is a finite dimensional complex vector space. Indeed, if we write $D = P - N$, with P and N nonnegative divisors with disjoint supports, then $\dim(L(D)) \leq 1 + \deg(P)$. In particular, if D is a nonnegative divisor, then

$$\dim(L(D)) \leq 1 + \deg(D).$$

Recall that a **meromorphic differential** on an open set $V \subseteq \mathbb{C}$ is an expression of the form $\omega = f(z)dz$, where f is a meromorphic function on V. Let $\omega_1 = f(z)dz$ and $\omega_2 = g(w)dw$ be two meromorphic differentials defined on V_1 and V_2. Let T be a holomorphic function from V_2 to V_1. We say that ω_1 **transforms** to ω_2 under T if

$$g(w) = f(T(w))T'(w).$$

If X is a Riemann surface, a **meromorphic** 1**-form on** X is a collection of meromorphic differentials (ω_φ), one for each chart $\varphi : U \longrightarrow V$ and defined on V, such that if two charts $\varphi_1 : U_1 \longrightarrow V_1$ and $\varphi_2 : U_2 \longrightarrow V_2$ have overlapping domains, then ω_{φ_1} transforms to ω_{φ_2} under the transition function $\varphi_1 \circ \varphi_2^{-1}$. The notion of **holomorphic** 1**-form** is defined in a similar way.

Given a meromorphic 1-form ω on X, for each p choose a local coordinate z_p centred at p. We may write $\omega = f(z_p)dz_p$, where f is a meromorphic function at $z = 0$. The **order of** p, denoted by $\text{ord}_p(\omega)$, is the order of the function f at $z_p = 0$. We can also write ω via the Laurent series of f in the coordinate z_p:

$$\omega = f(z_p)dz_p = \left(\sum_{n=-M}^{\infty} c_n z_p^n \right) dz_p$$

where $c_{-M} \neq 0$, so that $\mathrm{ord}_p(\omega) = -M$. The **residue of ω at p**, denoted by $\mathrm{Res}_p(\omega)$, is the coefficient c_{-1} in a Laurent series for ω at p. You can see the proof of the following theorem in [3, Theorem 3.17].

Theorem 1.1.1 (The Residue Theorem)**.** Let ω be a meromorphic 1-form on a compact Riemann surface X. Then

$$\sum_{p \in X} \mathrm{Res}_p(\omega) = 0.$$

The divisor of ω, denoted by (ω), is the divisor defined by the order function

$$(\omega) = \sum_p \mathrm{ord}_p(\omega) \cdot p.$$

Any divisor of this form is called a **canonical divisor**.

Proposition 1.1.2. If X is a compact Riemann surface of genus g which has a nonconstant meromorphic function, then there is a canonical divisor on X of degree $2g - 2$.

Given a divisor D on X, the **space of meromorphic 1-forms with poles bounded by** D, denoted $L^{(1)}(D)$, is the set of meromorphic 1-forms

$$L^{(1)}(D) = \{\omega \ / \ (\omega) \geq -D\}.$$

It is clear that $L^{(1)}(D)$ is a complex vector space. Note that $L^{(1)}(0) = \Omega(X)$, the space of holomorphic 1-forms.

The following result gives a way to relate the spaces $L(-)$ and $L^{(1)}(-)$. You can check the proof in [3, Lemma 3.11].

Proposition 1.1.3. Let D be a divisor on X, and let K be a canonical divisor on X. Then the spaces $L^{(1)}(D)$ and $L(D + K)$ are isomorphic.

A meromorphic function f on a Riemann surface X **has multiplicity one at a point** $p \in X$ if either f is holomorphic at $p \in X$ and $\mathrm{ord}_p(f - f(p)) = 1$, or f has a simple pole at p. Recall that for every meromorphic function $f : X \longrightarrow \mathbb{C}$, there corresponds a unique holomorphic map $F : X \longrightarrow \mathbb{C} \cup \{\infty\}$ which is not identically ∞. Such a map is given by

$$F(x) = \begin{cases} f(x) \in \mathbb{C} & \text{if } x \text{ is not a pole of } f, \\ \infty & \text{if } x \text{ is a pole of } f. \end{cases}$$

It is clear that f has multiplicity one at p if and only if F does.

Let S be a set of meromorphic functions on a compact Riemann surface X.

(1) We say that S **separates points in** X if for every pair of distinct points p and q in X there is a meromorphic function $f \in S$ such that $f(p) \neq f(q)$.

(2) We say that S **separates tangents of** X if for every point $p \in X$ there is a meromorphic function $f \in S$ which has multiplicity one at p.

A compact Riemann surface X is an **algebraic curve** if the field $\mathcal{M}(X)$ of meromorphic functions on X separates points and tangents of X.

Note that (1) is easy to understand. But (2) is not so clear. Suppose that S separates tangents. Then there is a meromorphic function $f : X \longrightarrow \mathbb{C}$ having multiplicity one at each $p \in X$. Then the associated holomorphic map $F : X \longrightarrow \mathbb{C} \cup \{\infty\}$ has multiplicity one at each $p \in X$. So $\text{ord}_p(F - F(p)) = 1$, i.e. $(F - F(p))'(p) \neq 0$. Hence the derivative map $F' : T_pX \longrightarrow T_{F(p)}\mathbb{C} \cup \{\infty\}$ is one-to-one. Note also that if $\text{ord}_p(f - f(p)) = 1$, then f is a local coordinate chart, since it is locally a biholomorphic map from an open neighbourhood $U \subseteq X$ to an open set $V \subseteq \mathbb{C}$. Therefore, S separates tangents if and only if at every point $p \in X$, there is a local coordinate which extends to a meromorphic function on all X. It follows that if X is an algebraic curve, then for every $p \in X$ there exists a meromorphic function g on X such that $\text{ord}_p(g) = 1$. For take a function f separating tangents at p, if f is holomorphic at p, set $g = f - f(p)$, and if f has a simple pole at p set $g = 1/f$.

Example 1.1.1. The Riemann sphere $\mathbb{C} \bigcup \{\infty\}$ and the complex torus $\mathbb{C}/\Gamma$ are examples of algebraic curves. It is known that every compact Riemann surface is an algebraic curve.

1.2 Laurent series approximation

In this section we prove the Laurent Series Approximation Theorem. By the comments in the previous section, we can find a meromorphic function g on X such that $\text{ord}_p(g) = 1$. Then if we set $f = g^N$, we get the following result.

Lemma 1.2.1. Let X be an algebraic curve, and let $p \in X$. Then for any integer N there is a global meromorphic function f on X with $\text{ord}_p(f) = N$.

A Laurent polynomial $r(z) = \sum_{i=n}^{m} c_i z^i$ is called a **Laurent tail** of a Laurent series $h(z)$ if the Laurent series starts with $r(z)$.

Lemma 1.2.2. Let X be an algebraic curve. Fix a point $p \in X$ and a local coordinate z centred at p. Fix any Laurent polynomial $r(z)$ in z. Then there exists a meromorphic function f on X whose Laurent series at p has $r(z)$ as a Laurent tail.

Proof: Let $r(z) = \sum_{i=n}^{m} c_i z^i$ be our Laurent polynomial, where $c_n, c_m \neq 0$. Note that $r(z)$ has $m - n + 1$ terms. The result follows by induction on the number of terms of $r(z)$. Suppose $m - n + 1 = 1$. Then $z(z) = cz^m$. By the previous lemma, there exists a meromorphic function f on X with $\text{ord}_p(f) = m$. Then its Laurent series in the coordinate z has the form $\sum_{i=m}^{\infty} a_i z^i$, where $a_m \neq 0$. Then $\frac{c}{a_m} f$ is the desired meromorphic function. Now assume that $m - n + 1 \geq 2$. We know that we can find a meromorphic function h having $c_n z^n$ as a Laurent tail. Consider the meromorphic function $h - r$. Let $s(z)$ be the Laurent polynomial which is the tail of the Laurent series of $h - r$ at p, up through the z^m term. Note that $h(z) - r(z)$ starts at $n + 1$. It is clear that $s(z)$ has fewer terms than $r(z)$. By induction, there is a meromorphic function g on X whose Laurent series at p has s as a tail. Now consider $f = h - g$. We have

$$f = (h - r) - g + r = \cdots + s(z) - \cdots - s(z) + r(z) = \cdots + r(z),$$

and so f has r as a tail. □

Lemma 1.2.3. Let X be an algebraic curve. Then for any finite number of points $p, q_1, \ldots, q_n$ in X, there is a meromorphic function f on X with a zero at p and a pole at each q_i.

Proof: We proceed by induction on n. For the case $n = 1$, let g be a meromorphic function on X such that $g(p) \neq g(q)$. Suppose that p is not a pole of g. We may assume that $g(p) = 0$. There is nothing to show if q is a pole of g. If not, then consider $f = g/(g(q) - g)$. Then $f(p) = 0$ and f has a pole at q. Now if p is a pole of g, repeat the previous argument with $1/g$.

Suppose $n \geq 2$ and let g be a meromorphic function with a zero at p and a pole at $q_1, \ldots, q_{n-1}$. On the other hand, there is a meromorphic function h with a zero at p and a pole at q_n. Consider the point q_1 and the function $g + h$. If h is holomorphic at q_1, then $g + h$ has a pole at q_1. If h has a pole at q_1, then it may occur that $g + h$ has not a pole at q_1. For m sufficiently large, $g + h^m$ has a pole at q_1 since the order of h^m at q_1 is greater than the order of g at the same point. Increasing m if necessary, we may assume that $g + h^m$ has a pole at each q_i with $i \leq n - 1$. We know that g has a pole at q_n, but it may occur that $g + h^m$ does not. In this case, increase the value of m. Thus, $f = g + h^m$ has a zero at p and a pole at $q_1, \ldots, q_n$ for a large m. □

Lemma 1.2.4. Let X be an algebraic curve. Then for any finite number of points $p, q_1, \ldots, q_n$ in X, and any $N \geq 1$, there is a global meromorphic function f on X with $\text{ord}_p(f - 1) \geq N$ and $\text{ord}_{q_i}(f) \geq N$ for each i.

Proof: By the previous lemma, there exists a meromorphic function g on X with a zero at p and a pole at each q_i. Consider the function $f = 1/(1 + g^N)$. We have $f(z) - 1 = -g^N/(1 + g^N)$. Then p is a zero of $f - 1$ of order at least N. It is easy to see that $\text{ord}_{q_i}(f) \geq N$. □

Theorem 1.2.1 (Laurent Series Approximation)**.** Suppose X is an algebraic curve. Fix a finite number of points $p_1, \ldots, p_n$ in X, choose a local coordinate z_i at each p_i, and finally choose Laurent polynomials $r_i(z_i)$ for each i. Then there is a meromorphic function f on X such that for every i, f has r_i as a Laurent tail at p_i.

Proof: By Lemma 1.2.2 there are meromorphic functions g_i on X such that g_i has r_i as a Laurent tail at p_i. Let

$$M = \min\{\text{ord}_{p_i}(r_i) \ / \ i = 1, \ldots, n\} = \min\{\text{ord}_{p_i}(g_i) \ / \ i = 1, \ldots, n\}.$$

Let N be a fixed integer larger than every exponent of every term of every r_i. By Lemma 1.2.4, there are meromorphic function h_i on X such that for each i, $\text{ord}_{p_i}(h^i - 1) \geq N - M$ and $\text{ord}_{p_j}(h_i) \geq N - M$ for $j \neq i$. The function $f = \sum_{i=1}^{n} h_i \cdot g_i$ has r_i as a Laurent tail at p_i, for each p_i. □

Corollary 1.2.1. Let X be an algebraic curve. Fix a finite number of points $p_1, \ldots, p_n$ in X, and a finite number of integers m_i. Then there exists a meromorphic function f on X such that $\text{ord}_{p_i}(f) = m_i$ for each i.

1.3 The function field $\mathcal{M}(X)$ as an extension of the field of rational expressions in some nonconstant meromorphic function

Let $\mathbb{C}(f)$ denote the field of rational expressions in f, where f is a nonconstant meromorphic function on an algebraic curve X. We have the inclusions of fields $\mathbb{C} \subseteq \mathbb{C}(f) \subseteq \mathcal{M}(X)$. Then $\mathcal{M}(X)$ is a field extension of $\mathbb{C}(f)$, and hence $\mathcal{M}(X)$ can be considered as a vector space over $\mathbb{C}(f)$. Let $[\mathcal{M}(X) : \mathbb{C}(f)]$ denote the dimension of $\mathcal{M}(X)$ over $\mathbb{C}(f)$, which is called the **degree of the extension** $\mathcal{M}(X)/\mathbb{C}(f)$. In this section we prove that $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg((f)_\infty)$. For this it is necessary the theorem of Laurent Series Approximation. But this result will only help to show the inequality $[\mathcal{M}(X) : \mathbb{C}(f)] \geq \deg((f)_\infty)$. We shall need other results to prove the other inequality.

Lemma 1.3.1. Let A be a divisor on a compact Riemann surface X, and let $D = (f)_\infty$ for a nonconstant meromorphic function f on X. Then there is an integer $m > 0$ and a meromorphic function g on X such that $A - (g) \leq mD$. Moreover, g can be taken to be a polynomial in f: $g = r(f)$ for some polynomial $r(t) \in \mathbb{C}[t]$.

Proof: Let $p_1, \ldots, p_k$ be points in the support of A such that

(i) $p_1, \ldots, p_k$ are not poles of f.

(ii) $A(p_i) \geq 1$ for each $1 \leq i \leq k$.

Then $f(p_i) \in \mathbb{C}$, and so we can consider $f - f(p_i)$ as a meromorphic function with a zero at p_i or order at least 1, and having the same poles as f. It follows that $(f - f(p_i))^{A(p_i)}$ is a meromorphic function with a zero at p_i of order at least $A(p_i)$, and whose poles are the poles of f. Let $g = \prod_{i=1}^k (f - f(p_i))^{A(p_i)}$. Then g is a meromorphic function such that $A - (g)$ is positive only at the poles of f. Now suppose that p is a pole of f, we have

$$\begin{aligned} A(p) - (g)(p) = A(p) - \mathrm{ord}_p(g) &= A(p) - \sum_{i=1}^k A(p_i)\mathrm{ord}_p(f - f(p_i)) \\ &= A(p) - \sum_{i=1}^k A(p_i)\mathrm{ord}_p(f) = A(p) - \left(\sum_{i=1}^k A(p_i)\right) \cdot \mathrm{ord}_p(f) \\ &= A(p) + \left(\sum_{i=1}^k A(p_i)\right) \cdot (f)_\infty(p) \\ &\leq m(f)_\infty, \text{ for some } m \text{ sufficiently large.} \end{aligned}$$

If p is not a pole of f, then we have that $A(p) - (g)(p) < 0 < m(f)_\infty(p)$. □

Corollary 1.3.1. Let X be a compact Riemann surface, and let f and h be nonconstant meromorphic functions on X. Then there is a polynomial $r(t) \in \mathbb{C}[t]$ such that the function $r(f)h$ has no poles outside of the poles of f. In this case there is an integer m such that $r(f)h \in L(mD)$, where $D = (f)_\infty$.

Proof: Put $A = -(h)$ in the previous lemma. There exists a meromorphic function g on X such that $-(h) - (g) \leq m(f)_\infty$, for some integer $m > 0$, where such a g can be chosen as a polynomial function in f, say $g = r(f)$. First, we show that $r(f)h \in L(mD)$, i.e. $(r(f)h) \geq -mD$. Let $p \in X$, we have $(r(f)h)(p) = (g)(p) + (h)(p) \geq -mD(p)$. Now we show that $r(f)h$ has no poles outside of the poles of f. Suppose $p \in X$ is a pole of $r(f)h$. Then $\text{ord}_p(r(f)h) < 0$. We have $0 > \text{ord}_p(r(f)h) \geq -m(f)_\infty(p) = m \cdot \text{ord}_p(f)$. Since $m > 0$, we have $\text{ord}_p(f) < 0$, i.e. p is a pole of f. □

Lemma 1.3.2. Fix a meromorphic function f on a compact Riemann surface X, and let $D = (f)_\infty$. Suppose that $[\mathcal{M}(X) : \mathbb{C}(f)] \geq k$. Then there is an integer m_0 such that for all $m \geq m_0$,

$$\dim(L(mD)) \geq (m - m_0 + 1)k.$$

Proof: Since $[\mathcal{M}(X) : \mathbb{C}(f)] \geq k$, we can choose linearly independent meromorphic functions $g_1, \ldots, g_k$ over $\mathbb{C}(f)$. By the previous corollary, for each i there exists a nonzero rational expression $r_i(f)$ such that $h_i = r_i(f)g_i$ has no poles outside of the poles of f. It is clear that the $h_1, \ldots, h_k$ are linearly independent over $\mathbb{C}(f)$. For each i there is an integer m_i such that $h_i \in L(m_i D)$. Let $p \in X$, we have $(h_i)(p) \geq -m_i D(p)$. Let $m_0 = \max\{m_i : i = 1, \ldots, k\}$. Then $(h_i)(p) \geq -m_0 D(p)$, for every i, that is $h_i \in L(m_0 D)$. Now we show that for any integer $m \geq m_0$, one has

$$\dim(L(mD)) \geq (m - m_0 + 1)k.$$

First, note that $f^i h_j \in L(mD)$ if $i \leq m - m_0$. For let $p \in X$, we have

$$\begin{aligned}(f^i h_j)(p) &= \text{ord}_p(f^i h_j) = \text{ord}_p(f^i) + \text{ord}_p(h_j) = i \cdot \text{ord}_p(f) + \text{ord}_p(h_j) \\ &\geq i \cdot \text{ord}_p(f) - m_0 D(p) = i \cdot \text{ord}_p(f) + m_0 \cdot \text{ord}_p(f) \\ &= (i + m_0) \cdot \text{ord}_p(f) = -(i + m_0)(f)_\infty(p) \\ &\geq -mD(p).\end{aligned}$$

Then $f^i h_j \in L(mD)$. Note that the $f^i h_j$'s are linearly independent over $\mathbb{C}(f)$. Then the $\mathbb{C}(f)$-vector space generated by the $f^i h_j$'s has dimension $(m - m_0 + 1)k$. It follows $\dim(L(mD)) \geq (m - m_0 + 1)k$. □

Now we are ready to compute $[\mathcal{M}(X) : \mathbb{C}(f)]$.

Theorem 1.3.1. Let f be a nonconstant meromorphic function on an algebraic curve X. Then $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg(D)$, where $D = (f)_\infty$.

Proof: We first show that $[\mathcal{M}(X) : \mathbb{C}(f)] \leq \deg(D)$. Suppose $[\mathcal{M}(X) : \mathbb{C}(f)]$ is at least $\deg(D)+1$. By the previous lemma, there exists an integer m_0 such that for all $m \geq m_0$,

$$\dim(L(mD)) \geq (m - m_0 + 1)(1 + \deg(D)).$$

On the other hand,

$$\dim(L(mD)) \leq 1 + \deg(mD) = 1 + m\deg(D).$$

So we get

$$\begin{aligned}(m - m_0 + 1)(1 + \deg(D)) &\leq 1 + m\deg(D)\\ m - m_0 - m_0\deg(D) + \deg(D) &\leq 0\\ m + \deg(D) &\leq m_0 + m_0\deg(D),\end{aligned}$$

for every m, getting a contradiction.

To show the other inequality, write $D = \sum n_i \cdot p_i$, with each $n_i \geq 1$, since D is a nonnegative divisor. By Corollary 1.2.1, for each p_i and each j there is a meromorphic function g_{ij} with a pole at p_i or order j, and no zero or pole at any of the other p_k's. The set $\{g_{ij} \ / \ 1 \leq j \leq n_i\}$ is linearly independent over $\mathbb{C}(f)$, for each i. Then we have

$$[\mathcal{M}(X) : \mathbb{C}(f)] \geq \sum_i \text{card}(\{g_{ij} \ / \ 1 \leq j \leq n_i\}) = \sum_i n_i = \deg(D).$$

□

Chapter 2

The Grothendieck-Riemann-Roch Theorem

In this chapter we shall prove the Riemann-Roch Theorem for algebraic curves, and comment two generalizations known as the Hirzebruch-Riemann-Roch Theorem and the Grothendieck-Riemann-Roch Theorem. In the first section we study certain sums called Laurent tail divisors. We can relate Laurent tail divisors and ordinary divisors via an operation called truncation. This allows us to define a group $\mathcal{T}[D](X)$, where X is an algebraic curve, formed by Laurent tail divisors that are bounded, in some sense, by a divisor D. We can also relate meromorphic functions to Laurent tail divisors. In fact, for every meromorphic function there corresponds an element in $\mathcal{T}[D]$ via a group homomorphism $\alpha_D : \mathcal{M}(X) \longrightarrow \mathcal{T}[D](X)$. This map is of vital importance to prove the Riemann-Roch Theorem. Recall that this theorem gives us a formula to compute the dimension of the space $L(D)$, which turns out to be the kernel of the homomorphism α_D. A first form of this theorem involves the cokernel of this homomorphism. At this point, the problem is that we are dealing with two spaces, namely $L(D)$ and the cokernel of α_D. We shall prove and use a result called Serre Duality to give an easy way two compute the dimension of the cokernel of α_D, and so we shall obtain a refined version of the Riemann-Roch Theorem that allows us to compute the dimension of $L(D)$ only in terms of the degree of D and the genus of X.

2.1 Laurent tail divisors

Let X be a compact Riemann surface. For each point $p \in X$, choose a local coordinate z_p centred at p. A **Laurent tail divisor on** X is a finite formal sum of the form

$$\sum_p r_p \cdot p,$$

where $r_p(z_p)$ is a Laurent polynomial in the coordinate z_p. The set of Laurent tail divisors forms a group under formal addition, denoted $\mathcal{T}(X)$. We shall focus on special subgroups of $\mathcal{T}(X)$: for any divisor D, define $\mathcal{T}[D](X)$ as the set of all finite formal sums $\sum_p r_p \cdot p$ such that for all p with $r_p \neq 0$, the top term of r_p has degree strictly less that $-D(p)$. Note that $\mathcal{T}[D](X)$ is a subgroup of $\mathcal{T}(X)$.

From this point, we shall construct group homomorphisms concerning the previous groups. Consider a Laurent tail divisor $\sum_p r_p \cdot p$ and a divisor D. At each p, we have

$$r_p(z_p) = \sum_{i=n_p}^{m_p} a_i^p z_p^i.$$

Let i_D be the smallest integer between n_p and m_p such that $i_D + 1 \geq -D(p)$. We can define a **truncation** of $r_p(z_p)$ as

$$\sum_{i=n_p}^{m_p} a_i^p z_p^i \mapsto \sum_{i=n_p}^{i_D} a_i^p z_p^i.$$

This mapping defines a group homomorphism

$$t_D : \mathcal{T}(X) \longrightarrow \mathcal{T}[D](X)$$

that sends each $\sum_p r_p \cdot p$ to $\sum_p \hat{r_p} \cdot p$, where $\hat{r_p}$ denotes the truncation of r_p whose top term is the largest integer between n_p and m_p strictly smaller than $-D(p)$, in other words, t_D is defined by removing from each $r_p(z_p)$ those terms of degree greater or equal than $-D(p)$.

Now suppose that we have two divisors D_1 and D_2 such that $D_1 \leq D_2$, i.e. $D_1(p) \leq D_2(p)$ for every $p \in X$. Then $-D_2(p) \leq -D_1(p)$ for every $p \in X$. Let $\sum_p r_p \cdot p \in \mathcal{T}[D_1](X)$. For each p with $r_p \neq 0$, consider the Laurent polynomial $r_p(z_p) = \sum_{i=n_p}^{m_p} a_i^p z_p^i$, where $m_p < -D_1(p)$. Suppose there exists an integer between n_p and m_p greater or equal than $-D_2(p)$, and let $i_{D_2} + 1$ be the smallest of such integers. Then we can truncate $r_p(z_p)$:

$$\sum_{i=n_p}^{m_p} a_i^p z_p^i \mapsto \sum_{i=n_p}^{i_{D_2}} a_i^p z_p^i.$$

If such an i_{D_2} does not exist, then just map $\sum_{i=n_p}^{m_p} a_i^p z_p^i$ to 0. This truncation defines a group homomorphism

$$t_{D_2}^{D_1} : \mathcal{T}[D_1](X) \longrightarrow \mathcal{T}[D_2](X)$$

defined by removing from each $r_p(z_p)$ those terms of degree greater or equal than $-D_2(p)$. We shall call the maps $t_{D_2}^{D_1}$ **truncation maps**.

Consider a meromorphic function f and a divisor D. Let $\sum_p r_p \cdot p$ be a Laurent tail in $\mathcal{T}[D](X)$. Let $\sum_{i=n_p}^{\infty} a_i z_p^i$ be the Laurent series of f in the coordinate z_p, and let $r_p(z_p) = \sum_{j=m_p}^{k} b_j z_p^j$. We take the product $\left(\sum_{i=n_p}^{\infty} a_i z_p^i\right) \cdot \left(\sum_{j=m_p}^{k} b_j z_p^j\right) = \sum_{i=n_p,\ j=m_p}^{\infty} a_i b_j z_p^{i+j}$, and truncate it by removing those terms of degree greater or equal than $-D(p) + (f)(p)$. This gives rise to a group homomorphism

$$\mu_f^D : \mathcal{T}[D](X) \longrightarrow \mathcal{T}[D - (f)](X)$$

mapping each $\sum_p r_p \cdot p$ to $\sum_p (fr_p) \cdot p$, where $f \cdot r_p$ is the Laurent polynomial of the above truncation of the series $\sum_{i=n_p,\ j=m_p}^{\infty} a_i b_j z_p^{i+j}$. It is straightforward to check that μ_f^D has an inverse $\mu_{1/f}^{D-(f)}$.

Now consider again the Laurent series of f: $\sum_{i=n_p}^{\infty} a_i z_p^i$. Given a divisor D, there exists a smallest integer m_p such that $m_p + 1 \geq -D(p)$. Then we can truncate the previous series and get a Laurent polynomial $r_p(z_p) = \sum_{i=n_p}^{m_p} a_i z_p^i$. This way we get a map

$$\alpha_D : \mathcal{M}(X) \longrightarrow \mathcal{T}[D](X)$$

which turns out to be a group homomorphism.

Proposition 2.1.1 (Properties of α_D).

(1) α_D commutes with the truncation maps: If D_1 and D_2 are divisors with $D_1 \leq D_2$ then the following triangle commutes:

$$\begin{array}{ccc} \mathcal{M}(X) & \xrightarrow{\alpha_{D_1}} & \mathcal{T}[D_1](X) \\ & \searrow^{\alpha_{D_2}} & \downarrow t_{D_2}^{D_1} \\ & & \mathcal{T}[D_2](X) \end{array}$$

(2) α_D is compatible with the multiplication operators: If f and g are meromorphic functions on X, then

$$\mu_f^D(\alpha_D(g)) = \alpha_{D-(f)}(f \cdot g)$$

for any divisor D.

(3) $L(D) = \mathrm{Ker}(\alpha_D)$.

Proof: Let $p \in X$ and let z_p be a coordinate centred at p.

(1) Let $\sum_{i=n_p}^{\infty} a_i z_p^i$ be the Laurent series of f in the coordinate z_p, and let m_p be the largest integer greater or equal than n_p such that $m_p < -D_1(p)$. Then $\alpha_D(f) = \sum_p r_p \cdot p$, where $r_p(z_p) = \sum_{i=n_p}^{m_p} a_i z_p^i$. Notice that $-D_2(p) \leq -D_1(p)$. Let k_p be

the largest integer greater or equal than n_p such that $k_p+1 \geq -D_2(p)$. Note that $k_p < m_p$. Then

$$t_{D_2}^{D_1} \circ \alpha_{D_1}(f) = \sum_{i=n_p}^{k_p} a_i z_p^i = \alpha_{D_2}(f).$$

(2) Consider the Laurent series $f(z_p) = \sum_{i=n_p}^{\infty} a_i z_i^p$ and $g(z_p) = \sum_{j=m_p}^{\infty} b_j z_p^j$ of f and g in the coordinate z_p. We have $(f \cdot g)(z_p) = \sum_{i=n_p,\ j=m_p}^{\infty} a_i b_j z_p^{i+j}$. On the one hand, $\alpha_D(g) = \sum_{j=m_p}^{k} b_j z_p^j$, where $k \geq m_p$ is the largest integer satisfying $k < -D(p)$. Map $\alpha_D(g)$ to the series $\left(\sum_{i=n_p}^{\infty} a_i z_p^i\right) \cdot \left(\sum_{j=m_p}^{k} b_j z_p^j\right) = \sum_{i=n_p}^{\infty} \sum_{j=m_p}^{k} a_i b_j z_p^{i+j}$. On the other hand, let $\overline{k} \geq m_p + n_p$ be the largest integer such that

$$\overline{k} < -D(p) + (f)(p) = -D(p) + \mathrm{ord}_p(f).$$

Then $\mu_f^D(\alpha_D(g)) = \sum_{i=n_p,\ j=m_p}^{\overline{k}} a_i b_j z_p^{i+j} = \alpha_{D-(f)}(f \cdot g)$.

(3) Let $f \in L(D)$. Then $(f) \geq -D$. Consider the Laurent series $f(z_p) = \sum_{i=n_p}^{\infty} a_i z_p^i$. Since $n_p = \mathrm{ord}_p(f) \geq -D(p)$, the previous series is mapped to 0 by α_D. Now let $f \in \mathrm{Ker}(\alpha_D)$. Then each $r_p(z_p) = 0$ and hence $n_p \geq -D(p)$, i.e. $(f)(p) \geq -D(p)$.

□

2.2 The first form of the Riemann-Roch Theorem

In the previous section, we proved that $\mathrm{Ker}(\alpha_D) = L(D)$. What could we say about $\mathrm{CoKer}(\alpha_D)$?. Define

$$H^1(D) := \mathrm{CoKer}(\alpha_D) = \mathcal{T}[D](X)/\mathrm{Im}(\alpha_D)$$

We shall prove that the space $H^1(D)$ is finite dimensional. By (3) of the previous proposition, we have a short exact sequence

$$0 \longrightarrow L(D) \longrightarrow \mathcal{M}(X) \xrightarrow{\alpha_D} \mathcal{T}[D](X) \longrightarrow H^1(D) \longrightarrow 0$$

Consider the quotient space $\mathcal{M}(X)/L(D)$. Since α_D vanishes on $L(D)$, we can consider the map

$$\overline{\alpha_D} : \mathcal{M}(X)/L(D) \longrightarrow \mathcal{T}[D](X)$$

given by $\overline{\alpha_D}(f + L(D)) = \alpha_D(f)$. It is clear that it is a well defined group monomorphism. Also, note that $\mathrm{Im}(\overline{\alpha_D}) = \mathrm{Im}(\alpha_D) = \mathrm{Ker}(\mathcal{T}[D](X) \longrightarrow H^1(D))$. Then we get the following short exact sequence:

$$0 \longrightarrow \mathcal{M}(X)/L(D) \longrightarrow \mathcal{T}[D](X) \longrightarrow H^1(D) \longrightarrow 0$$

Now let D_1 and D_2 be two divisors satisfying $D_1 \leq D_2$. Then we have a truncation map $t_{D_2}^{D_1} : \mathcal{T}[D_1](X) \longrightarrow \mathcal{T}[D_2](X)$. Also, $L(D_1) \subseteq L(D_2)$. For each D_i there is a short exact sequence as above. We shall connect these sequences constructing two homomorphisms.

(i) Let $F : \mathcal{M}(X)/L(D_1) \longrightarrow \mathcal{M}(X)/L(D_2)$ be the map given by

$$F(f + L(D_1)) = f + L(D_2).$$

This map is well defined. For if $f - g \in L(D_1)$ then $(f) - (g) \geq -D_1 \geq -D_2$, i.e. $f - g \in L(D_2)$. Also, it is clear that F is a group homomorphism.

(ii) Now define a map $G : H^1(D_1) \longrightarrow H^1(D_2)$ by

$$G(Z + \mathrm{Im}(\alpha_{D_1})) = t_{D_2}^{D_1}(Z) + \mathrm{Im}(\alpha_{D_2}).$$

We check G is well defined. Suppose $Z - Z' \in \mathrm{Im}(\alpha_{D_1})$. Then $Z - Z' = \alpha_{D_1}(f)$ for some meromorphic function f. By the first part of the previous proposition, we have

$$t_{D_2}^{D_1}(Z) - t_{D_2}^{D_1}(Z') = t_{D_2}^{D_1} \circ \alpha_{D_1}(f) = \alpha_{D_2}(f) \in \mathrm{Im}(\alpha_{D_2})$$

It is clear that G is a group homomorphism.

Proposition 2.2.1. The following diagram commutes and has exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}(X)/L(D_1) & \xrightarrow{\overline{\alpha_{D_1}}} & \mathcal{T}[D_1](X) & \xrightarrow{\gamma_{D_1}} & H^1(D_1) & \longrightarrow & 0 \\
 & & \downarrow F & & \downarrow t_{D_2}^{D_1} & & \downarrow G & & \\
0 & \longrightarrow & \mathcal{M}(X)/L(D_2) & \xrightarrow[\overline{\alpha_{D_2}}]{} & \mathcal{T}[D_2](X) & \xrightarrow[\gamma_{D_2}]{} & H^1(D_2) & \longrightarrow & 0
\end{array}$$

where γ_{D_1} and γ_{D_2} are the projection maps.

Proof: On the one hand

$$\begin{aligned}
t_{D_2}^{D_1} \circ \overline{\alpha_{D_1}}(f + L(D_1)) &= t_{D_2}^{D_1}(\alpha_{D_1}(f)) = \alpha_{D_2}(f), \text{ by (1) of the previous proposition} \\
&= \overline{\alpha_{D_2}}(f + L(D_2)) \\
&= \overline{\alpha_{D_2}} \circ F(f + L(D_1)).
\end{aligned}$$

So the left square commutes. On the other hand,

$$G \circ \gamma_{D_1}(Z) = G(Z + \mathrm{Im}(\alpha_{D_1})) = t^{D_1}_{D_2}(Z) + \mathrm{Im}(\alpha_{D_2}) = \gamma_{D_2}(t^{D_1}_{D_2}(Z)).$$

Hence the right square commutes. $\square$

Let $H^1(D_1/D_2) := \mathrm{Ker}(G)$. We use this diagram two show that $H^1(D_1/D_2)$ is finite dimensional. This fact shall help us to prove the finite dimensionality of $H^1(D)$. Using the diagram above and the Snake Lemma, we obtain the following exact sequence

$$0 \longrightarrow \mathrm{Ker}(F) \longrightarrow \mathrm{Ker}(t^{D_1}_{D_2}) \longrightarrow \mathrm{Ker}(G) \longrightarrow \mathrm{CoKer}(F) \longrightarrow \mathrm{CoKer}(t^{D_1}_{D_2}) \longrightarrow \mathrm{CoKer}(G) \longrightarrow 0.$$

Note that F is surjective. We shall see that so are $t^{D_1}_{D_2}$ and G. Let $Z = \sum_p r_p \cdot p \in \mathcal{T}[D_2](X)$, where $r_p(z_p) = \sum_{i=n_p}^{m_p} a^p_i z^i_p$ and $m_p \geq n_p$ is the largest integer with $m_p < -D_2(p) \leq -D_1(p)$. So $Z \in \mathcal{T}[D_1](X)$ and $Z = t^{D_1}_{D_2}(Z)$. Hence $t^{D_1}_{D_2}$ is surjective, and this implies that so is G.

On the other hand, note that $\mathrm{Ker}(F) = L(D_2)/L(D_1)$. So we have

$$\dim(\mathrm{Ker}(F)) = \dim(L(D_2)) - \dim(L(D_1))$$

Now consider $\sum_p r_p \cdot p \in \mathrm{Ker}(t^{D_1}_{D_2})$, where $r_p(z_p) = \sum_{i=n_p}^{m_p} a^p_i z^i_p$. Since $\sum_p a^p_i z^i_p$ is mapped to 0, we have $n_p \geq -D_2(p)$. Hence $\mathrm{Ker}(t^{D_1}_{D_2})$ is the space of all $\sum_p r_p \cdot p$ such that the top term of r_p has order less than $-D_1(p)$ and the bottom term has order at least $-D_2(p)$. At each p, we have $D_2(p) - D_1(p)$ possible monomials z^i_p, $-D_2(p) \leq i \leq -D_1(p)$, that are linearly independent. Hence

$$\dim(\mathrm{Ker}(t^{D_1}_{D_2})) = \sum_p (D_2(p) - D_1(p)) = \sum_p D_2(p) - \sum_p D_1(p) = \deg(D_2) - \deg(D_1)$$

Recall that we have a short exact sequence

$$0 \longrightarrow L(D_2)/L(D_1) \longrightarrow \mathrm{Ker}(t^{D_1}_{D_2}) \longrightarrow H^1(D_1/D_2) \longrightarrow 0$$

Since $H^1(D_1/D_2)$ is a free $\mathbb{C}$-module, the previous sequence splits, so

$$\mathrm{Ker}(t^{D_1}_{D_2}) \cong (L(D_2)/L(D_1)) \bigoplus H^1(D_1/D_2)$$

It follows

$$\dim(\mathrm{Ker}(t^{D_1}_{D_2})) = \dim(L(D_2)/L(D_1)) + \dim(H^1(D_1/D_2))$$
$$\dim(H^1(D_1/D_2)) = \deg(D_2) - \deg(D_1) + \dim(L(D_1)) - \dim(L(D_2)).$$

We have proven that $H^1(D_1/D_2)$ is finite dimensional. Summarizing, we have

Lemma 2.2.1. If D_1 and D_2 are divisors on a compact Riemann surface X, with $D_1 \leq D_2$, then

$$\dim(H^1(D_1/D_2)) = [\deg(D_2) - \dim(L(D_2))] - [\deg(D_1) - \dim(L(D_1))]$$

As we commented above, we shall use the previous formula to prove the following result:

Proposition 2.2.2. For any divisor D on an algebraic curve X, $H^1(D)$ is a finite dimensional vector space over $\mathbb{C}$.

Before giving a proof, we need some lemmas.

Lemma 2.2.2. Let f be a nonconstant meromorphic function on an algebraic curve X, and let $D = (f)_\infty$. Then for any large m, the dimension of $H^1(0/mD)$ is constant, independent of m.

Proof: First, apply the previous lemma putting $D_1 = 0$ and $D_2 = mD$:

$$\begin{aligned}\dim(H^1(0/mD)) &= [\deg(mD) - \dim(L(mD))] - [\deg(0) - \dim(L(0))] \\ &= \deg(mD) - \dim(L(mD)) + 1.\end{aligned}$$

Note that we have used the equality $\dim(L(0)) = 1$, where $L(0)$ is the space of holomorphic functions on X. Since X is compact, we have that every holomorphic function on X is constant and hence $L(0) \cong \mathbb{C}$ has dimension one. Since $[\mathcal{M}(X) : \mathbb{C}(f)] = \deg(D)$ by Theorem 1.3.1, by Lemma 1.3.2 there is an integer m_0 such that

$$\dim(L(mD)) \geq (m - m_0 + 1)\deg(D)$$

for all $m \geq m_0$. Then we have

$$\begin{aligned}\dim(H^1(0/mD)) &= \deg(mD) - \dim(L(mD)) + 1 \\ &\leq \deg(mD) - (m - m_0 + 1)\deg(D) + 1 \\ &= (m_0 - 1)\deg(D) + 1.\end{aligned}$$

Considering $\dim(H^1(0/mD))$ as a function of m, we have it is bounded. We check it is nondecreasing. For suppose $0 < m_1 < m_2$. Then $0 < m_1 D < m_2 D$. We have surjective maps $H^1(0) \xrightarrow{G_{1,0}} H^1(m_1 D)$ and $H^1(m_1 D) \xrightarrow{G_{2,1}} H^1(m_2 D)$. Note that

$$G_{2,1} \circ G_{1,0} = G_{2,0} : H^1(0) \longrightarrow H^1(m_2 D).$$

It is easy to see that $\mathrm{Ker}(G_{1,0}) \subseteq \mathrm{Ker}(G_{2,0})$. Then $H^1(0/m_1 D) \subseteq H^1(0/m_2 D)$ and hence

$$\dim(H^1(0/m_1 D)) \leq \dim(H^1(0/m_2 D)).$$

We have that $\dim(H^1(0/mD))$ is a nondecreasing bounded function of m. Therefore, for m sufficiently large, we have $H^1(0/mD)$ is constant. □

Lemma 2.2.3. For any algebraic curve X and for any divisor A on X, there is an integer M such that

$$\deg(A) - \dim(L(A)) \leq M.$$

Proof: Fix a meromorphic function f and let $D = (f)_\infty$. Note that

$$\deg(mD) - \dim(L(mD)) = \dim(H^1(0/mD)) - 1 \leq M,$$

since $\dim(H^1(0/mD))$ is bounded. We know by Lemma 1.3.1 that there is a meromorphic function g on X and an integer m such that $B = A - (g) \leq mD$. We shall see A and B have the same degree. Since g is a nonconstant meromorphic function on a compact Riemann surface X, we have

$$\deg((g)) = \sum_p (g)(p) = \sum_p \operatorname{ord}_p(g) = 0.$$

Then we get

$$\deg(B) = \deg(A - (g)) = \deg(A) - \deg((g)) = \deg(A).$$

Now we show that $L(A) \cong L(B)$. If $f \in L(A)$ then $(f) \geq -A = -B - (g)$. We have

$$\begin{aligned} \operatorname{ord}_p(f) &\geq -B(p) - \operatorname{ord}_p(g) \\ \operatorname{ord}_p(f) + \operatorname{ord}_p(g) &\geq -B(p) \\ \operatorname{ord}_p(f \cdot g) &\geq -B(p). \end{aligned}$$

Then we can define a map $L(A) \longrightarrow L(B)$ by $f \mapsto f \cdot g$, which is clearly an isomorphism of vector spaces, whose inverse is given by $f \mapsto f/g$. It follows $\dim(L(A)) = \dim(L(B))$. On the other hand, by Lemma 2.2.1, $B \leq mD$ implies

$$\dim(H^1(B/mD)) = [\deg(mD) - \dim(L(mD))] - [\deg(B) - \dim(L(B))].$$

Hence, we have

$$\begin{aligned}\deg(A) - \dim(L(A)) &= \deg(B) - \dim(L(B)) \\ &= \deg(mD) - \dim(L(mD)) - \dim(H^1(B/mD)) \\ &\leq \deg(mD) - \dim(L(mD)) \\ &\leq M.\end{aligned}$$

□

Note the previous lemma implies that there exists a divisor A_0 on X such that the difference $\deg(A_0) - \dim(L(A_0))$ is maximal.

Lemma 2.2.4. $H^1(A_0) = 0$.

Proof: Suppose that $H^1(A_0) = \mathcal{T}[A_0](X)/\mathrm{Im}(\alpha_{A_0}) \neq 0$. Then there exists a Laurent tail divisor Z such that $Z \notin \mathrm{Im}(\alpha_{A_0})$. Let B be a divisor with $B \geq A_0$. Recall that $Z = \sum_p r_p \cdot p \in \mathrm{Ker}(t_B^{A_0})$ if and only if the top term of each r_p has order smaller than $-A_0(p)$ and the bottom term has order at least $-B(p)$. Choose a divisor B such that $Z \in \mathrm{Ker}(t_B^{A_0})$. Hence, $t_B^{A_0}(Z) + \mathrm{Im}(\alpha_B) = 0 + \mathrm{Im}(\alpha_B)$. So

$$Z + \mathrm{Im}(\alpha_{A_0}) \in \mathrm{Ker}(H^1(A_0) \xrightarrow{G_{B,A_0}} H^1(B)) = H^1(A_0/B).$$

Hence $H^1(A_0/B) \neq 0$ and $\dim(H^1(A_0/B)) \geq 1$. By Lemma 2.2.1, we get

$$1 \leq \dim(H^1(A_0/B)) = [\deg(B) - \dim(L(B))] - [\deg(A_0) - \dim(L(A_0))].$$

On the other hand,

$$[\deg(B) - \dim(L(B))] - [\deg(A_0) - \dim(L(A_0))] < 0$$

since $\deg(A_0) - \dim(L(A_0))$ is maximal, getting a contradiction. □

Proof of the Proposition 2.2.2: Write $D - A_0 = P - N$, where P and N are non-negative divisors. We have that $A_0 \leq A_0 + P$ and so we have a surjective map $H^1(A_0) \longrightarrow H^1(A_0 + P)$, where $H^1(A_0) = 0$. It follows $H^1(A_0 + P) = 0$. Since $A_0 + P - N \leq A_0 + P$, we have a short exact sequence

$$0 \longrightarrow H^1(A_0 + P - N/A_0 + P) \longrightarrow H^1(A_0 + P - N) \longrightarrow H^1(A_0 + P) \longrightarrow 0,$$

where $H^1(A_0 + P) = 0$. By exactness, we get

$$H^1(D) = H^1(A_0 + P - N) \cong H^1(A_0 + P - N/A_0 + P),$$

where the last space is finite dimensional by Lemma 2.2.1. □

So far we have proven that for any two divisors D_1 and D_2 with $D_1 \leq D_2$, there is a short exact sequence

$$0 \longrightarrow H^1(D_1/D_2) \longrightarrow H^1(D_1) \longrightarrow H^1(D_2) \longrightarrow 0$$

of finite dimensional vector spaces, where $H^1(D_1) \cong H^1(D_1/D_2) \bigoplus H^1(D_2)$ since this sequence splits. It follows

$$\dim(H^1(D_1/D_2)) = \dim(H^1(D_1)) - \dim(H^1(D_2)).$$

By Lemma 2.2.1, we also have

$$\dim(H^1(D_1/D_2)) = [\deg(D_2) - \dim(L(D_2))] - [\deg(D_1) - \dim(L(D_1))].$$

So we get

$$\begin{aligned} \deg(D_2) - \dim(L(D_2)) - \deg(D_1) + \dim(L(D_1)) &= \dim(H^1(D_1)) - \dim(H^1(D_2)) \\ \dim(L(D_1)) - \deg(D_1) - \dim(H^1(D_1)) &= \dim(L(D_2)) - \deg(D_2) - \dim(H^1(D_2)). \end{aligned}$$

Now let D_1 and D_2 be any divisors on X, and let D be a common maximum of D_1 and D_2. Then

$$\begin{aligned} \dim(L(D_1)) - \deg(D_1) - \dim(H^1(D_1)) &= \dim(L(D)) - \deg(D) - \dim(H^1(D)) \\ &= \dim(L(D_2)) - \deg(D_2) - \dim(H^1(D_2)). \end{aligned}$$

It follows that the integer $\dim(L(D)) - \deg(D) - \dim(H^1(D))$ is constant and it is equal to

$$\dim(L(0)) - \deg(0) - \dim(H^1(0)) = 1 - \dim(H^1(0)).$$

Summarizing, we have

Theorem 2.2.1 (Riemann-Roch: First Form)**.** Let D be a divisor on an algebraic curve X. Then

$$\dim(L(D)) - \dim(H^1(D)) = \deg(D) + 1 - \dim(H^1(0)).$$

2.3 Serre Duality and the Riemann-Roch Theorem

The First Form of the Riemann-Roch Theorem have some problems. In its formula, we need to consider three spaces: $L(D)$, $H^1(D)$ and $H^1(0)$. Our next goal is to find more suitable expressions for $\dim(H^1(D))$ and $\dim(H^1(0))$. Such expressions will be consequences of a result called Serre Duality. Roughly speaking, Serre duality states that the space of meromorphic 1-forms with poles bounded by $-D$, $L^{(1)}(-D)$, and the dual space to $H^1(D)$ are isomorphic. We first construct a linear map $L^{(1)}(-D) \longrightarrow H^1(D)^*$, which we shall call the **Residue map**. Suppose D is a divisor on X and ω a meromorphic 1-form on X in the space $L^{(1)}(-D)$, i.e. $\mathrm{ord}_p(\omega) \geq D(p)$ for all $p \in X$. It follows we can write

$$\omega = \left(\sum_{n=D(p)}^{\infty} c_n z_p^n \right) dz_p$$

in a local coordinate z_p about p, for each p. We define the following linear map:

$$\begin{aligned} \mathrm{Res}_\omega : \mathcal{T}[D](X) &\longrightarrow \mathbb{C} \\ \sum_p r_p \cdot p &\mapsto \sum_p \mathrm{Res}_p(r_p \cdot \omega). \end{aligned}$$

Suppose f is a meromorphic function on X. Write $f = \sum_k a_k z_p^k$ in the coordinate z_p. Near p, we have

$$f\omega = \left(\sum_k a_k z_p^k \right) \cdot \left(\sum_{n=D(p)}^{\infty} c_n z_p^n \right) dz_p.$$

The coefficient of $1/z_p$ is given by $\sum_{n=D(p)}^{\infty} c_n a_{-n-1}$. So

$$\mathrm{Res}_p(f\omega) = \sum_{n=D(p)}^{\infty} c_n a_{-n-1}.$$

The expression $\mathrm{Res}_p(f\omega)$ depends only on the Laurent tail divisor $\alpha_D(f) = \sum_p r_p \cdot p$, since $\sum_{n=D(p)}^{\infty} c_n a_{-n-1}$ depends on the coefficients a_i for f with $i < -D(p)$. Then we have

$$\mathrm{Res}_\omega(\alpha_D(f)) = \sum_p \mathrm{Res}_p(f\omega).$$

By the Residue Theorem, we get $\mathrm{Res}_\omega(\alpha_D(f)) = 0$. This means that the previous map Res_ω descends to a map

$$\mathrm{Res}_\omega : H^1(D) \longrightarrow \mathbb{C}$$

in $H^1(D)^*$. Therefore, we obtain a linear map

$$\begin{aligned} \mathrm{Res} : L^{(1)}(-D) &\longrightarrow H^1(D)^* \\ \omega &\mapsto \mathrm{Res}_\omega. \end{aligned}$$

Theorem 2.3.1 (Serre Duality)**.** For any divisor D on an algebraic curve X, the map

$$\mathrm{Res} : L^{(1)}(-D) \longrightarrow H^1(D)^*$$

is an isomorphism of complex vector spaces.

Before proving this theorem, we need two lemmas and some properties.

Proposition 2.3.1.

(1) If $\phi : \mathcal{T}[D](X) \longrightarrow \mathbb{C}$ is a linear functional that vanishes on $\alpha_D(\mathcal{M}(X))$, and f is any meromorphic function on X, then $\phi \circ \mu_f^{D+(f)} : \mathcal{T}[D+(f)](X) \longrightarrow \mathbb{C}$ is also a linear map vanishing on $\alpha_{D+(f)}(\mathcal{M}(X))$.

(2) Res_ω is compatible with the multiplication map μ_f: Suppose f is a meromorphic function on X and $\omega \in L^{(1)}(-D)$. Then $f\omega \in L^{(1)}(-D-(f))$ and

$$\mathrm{Res}_\omega \circ \mu_f^{D+(f)} = \mathrm{Res}_{f\omega}$$

as functionals on $\mathcal{T}[D+(f)](X)$

Proof: By the first part of Proposition 2.1.1, we have

$$\phi(\mu_f^{D+(f)}(\alpha_{D+(f)}(g))) = \phi(\alpha_{D+(f)-(f)}(f \cdot g)) = \phi(\alpha_D(f \cdot g)) = 0.$$

Hence (1) holds. Part (2) follows by noticing that

$$\mathrm{Res}_p((fr_p) \cdot \omega) = \mathrm{Res}_p(r_p \cdot (f\omega)).$$

□

Lemma 2.3.1. Suppose that ϕ_1 and ϕ_2 are two linear functionals on $H^1(A)$, for some divisor A. Then there is a positive divisor C and nonzero meromorphic functions f_1 and f_2 in $L(C)$ such that

$$\phi_1 \circ t_A^{A-C-(f_1)} \circ \mu_{f_1} = \phi_2 \circ t_A^{A-C-(f_2)} \circ \mu_{f_2}$$

as functionals on $H^1(A-C)$. In other words, the two maps on $\mathcal{T}[A-C](X)$ in the diagram

$$\begin{array}{ccccccc}
 & & \mathcal{T}[A-C-(f_1)](X) & \xrightarrow{t_A^{A-C-(f_1)}} & \mathcal{T}[A](X) & & \\
 & \overset{\mu_{f_1}}{\nearrow} & & & & \overset{\phi_1}{\searrow} & \\
\mathcal{T}[A-C](X) & & & & & & \mathbb{C} \\
 & \underset{\mu_{f_2}}{\searrow} & & & & \underset{\phi_2}{\nearrow} & \\
 & & \mathcal{T}[A-C-(f_2)](X) & \xrightarrow[t_A^{A-C-(f_2)}]{} & \mathcal{T}[A](X) & &
\end{array}$$

are equal for some C and some $f_1, f_2 \in L(C)\backslash\{0\}$.

Proof: First, note that ϕ_1 and ϕ_2 can be considered as linear functionals on $\mathcal{T}[A](X)$ that vanish on $\alpha_A(\mathcal{M}(X))$. Suppose no such divisor C and functions f_1 and f_2 exist. Then for every positive divisor C, we have a linear map

$$\begin{aligned} L(C) \times L(C) &\longrightarrow H^1(A-C) \\ (f_1, f_2) &\mapsto \phi_1 \circ t_A^{A-C-(f_1)} \circ \mu_{f_1} - \phi_2 \circ t_A^{A-C-(f_2)} \circ \mu_{f_2} \end{aligned}$$

which turns out to be injective. It follows

$$(1) \quad \dim(H^1(A-C)) \geq \dim(L(C) \times L(C)) = 2\dim(L(C)).$$

By the First Form of the Riemann-Roch Theorem applied to $A-C$ and C, we get

$$\begin{aligned}
\dim(L(A-C)) - \dim(H^1(A-C)) &= \deg(A-C) + 1 - \dim(H^1(0)) \\
\dim(H^1(A-C)) &= \dim(L(A-C)) - \deg(A-C) - 1 + \dim(H^1(0)) \\
(2) \quad \dim(H^1(A-C)) &\leq \dim(L(A)) - \deg(A) - 1 + \dim(H^1(0)) + \deg(C), \\
\dim(L(C)) - \deg(H^1(C)) &= \deg(C) + 1 - \dim(H^1(0)) \\
\dim(L(C)) &= \dim(H^1(C)) + \deg(C) + 1 - \dim(H^1(0)) \\
(3) \quad \dim(L(C)) &\geq \deg(C) + 1 - \dim(H^1(0)).
\end{aligned}$$

Connecting (1), (2) and (3), we have

$$\begin{aligned}
2\deg(C) + 2 - 2\dim(H^1(0)) &\leq 2\dim(L(C)) \\
&\leq \dim(H^1(A - C)) \\
&\leq \dim(L(A)) - \deg(A) - 1 + \dim(H^1(0)) + \deg(C) \\
2\deg(C) + 2 - 2\dim(H^1(0)) &\leq \dim(L(A)) - \deg(A) - 1 + \dim(H^1(0)) + \deg(C) \\
\deg(C) + 3 - 3\dim(H^1(0)) &\leq \dim(L(A)) - \deg(A) \\
\deg(C) &\leq \dim(L(A)) - \deg(A) - 3 + 3\dim(H^1(0)).
\end{aligned}$$

We have that $\deg(C)$ is bounded. Since C is arbitrary and positive, we can take the limit $\deg(C) \longrightarrow \infty$, getting a contradiction. □

Lemma 2.3.2. Suppose that D_1 is a divisor on X with $\omega \in L^{(1)}(-D_1)$, so that $\mathrm{Res}_\omega : \mathcal{T}[D_1](X) \longrightarrow \mathbb{C}$ is well defined. Suppose that $D_2 \geq D_1$ and that Res_ω vanishes on the kernel of $t^{D_1}_{D_2} : \mathcal{T}[D_1](X) \longrightarrow \mathcal{T}[D_2](X)$. Then $\omega \in L^{(1)}(-D_2)$.

Proof: Suppose $\omega \notin L^{(1)}(-D_2)$. Then there is a point $p \in X$ with $k = \mathrm{ord}_p(\omega) < D_2(p)$. Consider $Z = z_p^{-k-1} \cdot p$, which is a Laurent tail divisor in $\mathcal{T}[D_1](X)$ since $-k-1 < -D_1(p)$. On the other hand, $-k-1 \geq -D_2(p)$, and so $Z \in \mathrm{Ker}(t^{D_1}_{D_2})$.
Write $\omega = \left(\sum_{n=D_1(p)}^{\infty} c_n z_p^n\right) dz_p$. Then

$$z_p^{-k-1} \cdot \omega = \left(\sum_{n=D_1(p)}^{\infty} c_n z_p^{n-k-1}\right) dz_p.$$

Since $k = \mathrm{ord}_p(\omega)$, we have $\mathrm{Res}_\omega(Z) = \mathrm{Res}_p(z_p^{-k-1} \cdot \omega) = c_k \neq 0$, getting a contradiction. □

Proof of Serre Duality Theorem:

(1) Res is injective: Suppose the converse, i.e. that there exists $\omega \in L^{(1)}(-D)\backslash\{0\}$ such that $\text{Res}(\omega) \equiv 0$ on $H^1(D)$. Then $\sum_p \text{Res}_p(r_p \cdot \omega) = 0$ for every $\sum r_p \cdot p \in \mathcal{T}[D](X)$. Now fix $p \in X$ and a coordinate z_p about p. Note that $k = \text{ord}_p(\omega) \geq D(p)$ since $\omega \in L^{(1)}(-D)$. We have $-k-1 < D(p)$ and so $z_p^{-k-1} \cdot p \in \mathcal{T}[D](X)$. Write

$$\omega = \left(\sum_{n=k}^{\infty} c_n z_p^n \right) dz_p$$

where $c_k \neq 0$. We have

$$\text{Res}_\omega(z_p^{-k-1} \cdot p) = \text{Res}_p \left(z_p^{-k-1} \cdot \sum_{n=k}^{\infty} c_n z_p^n \right) = c_k \neq 0,$$

contradicting the fact that $\text{Res}_\omega \equiv 0$.

(2) Res is surjective: Let $\phi : H^1(D) \longrightarrow \mathbb{C}$ be a functional in $H^1(D)^*$, which can be considered as a functional $\mathcal{T}[D](X) \longrightarrow \mathbb{C}$ vanishing on $\alpha_D(\mathcal{M}(X))$. Let $\omega \neq 0$ be a meromorphic 1-form, and consider the canonical divisor $K = (\omega)$. We can find a divisor A such that $A \leq D$ and $A \leq K$. Then $\omega \in L^{(1)}(-A)$. Consider the composite map $\phi_A : \mathcal{T}[A](X) \xrightarrow{t_D^A} \mathcal{T}[D](X) \xrightarrow{\phi} \mathbb{C}$. We have two linear functionals on $\mathcal{T}[A](X)$, namely ϕ_A and Res_ω. Note that Res_ω vanishes on $\alpha_A(\mathcal{M}(X))$ since $\omega \in L^{-1}(-A)$. On the other hand,

$$\phi_A(\alpha_A(f)) = \phi \circ t_D^A \circ \alpha_A(f) = \phi \circ \alpha_D(f) = 0.$$

So ϕ_A also vanishes on $\alpha_A(\mathcal{M}(X))$. Now we can apply Lemma 2.3.1 to get a positive divisor C and meromorphic functions $f_1, f_2 \in L(C)$ such that

$$\phi_A \circ t_A^{A-C-(f_1)} \circ \mu_{f_1} = \text{Res}_\omega \circ t_A^{A-C-(f_2)} \circ \mu_{f_2}$$

where both sides can be considered as functionals on $H^1(A-C)$. For

$$\begin{aligned} \phi_A \circ t_A^{A-C-(f_1)} \circ \mu_{f_1}(\alpha_{A-C}(g)) &= \phi_A \circ t_A^{A-C-(f_1)} \circ \alpha_{A-C-(f_1)}(f_1 \cdot g) \\ &= \phi \circ t_D^A \circ \alpha_A(f_1 \cdot g) \\ &= \phi \circ \alpha_D(f_1 \cdot g) = 0. \end{aligned}$$

A similar argument shows that $\text{Res}_\omega \circ t_A^{A-C-(f_2)} \circ \mu_{f_2}$ also vanishes on $\alpha_{A-C}(\mathcal{M}(X))$. We shall prove that $\phi = \text{Res}((f_2/f_1)\omega)$. Since $-C \leq (f_2)$, we have $A-C \leq A+(f_2)$.

Then we can consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}[A-C](X) & \xrightarrow{\mu_{f_2}^{A-C}} & \mathcal{T}[A-C-(f_2)](X) \\ \downarrow t_{A+(f_2)}^{A-C} & & \downarrow t_A^{A-C-(f_2)} \\ \mathcal{T}[A+(f_2)](X) & \xrightarrow[\mu_{f_2}^{A+(f_2)}]{} & \mathcal{T}[A](X) \end{array}$$

Then we have

$$\begin{aligned} \mathrm{Res}_\omega \circ t_A^{A-C-(f_2)} \circ \mu_{f_2}^{A-C} &= \mathrm{Res}_\omega \circ \mu_{f_2}^{A+(f_2)} \circ t_{A+(f_2)}^{A-C} \\ \phi_A \circ t_A^{A-C-(f_1)} \circ \mu_{f_1}^{A-C} &= \mathrm{Res}_\omega \circ \mu_{f_2}^{A+(f_2)} \circ t_{A+(f_2)}^{A-C} = \mathrm{Res}_{f_2\omega} \circ t_{A+(f_2)}^{A-C} \\ \phi_A \circ t_A^{A-C-(f_1)} &= \mathrm{Res}_{f_2\omega} \circ t_{A+(f_2)}^{A-C} \circ \mu_{1/f_1}^{A-C-(f_1)}. \end{aligned}$$

Similarly, we have another commutative square

$$\begin{array}{ccc} \mathcal{T}[A-C-(f_1)](X) & \xrightarrow{\mu_{1/f_1}^{A-C-(f_1)}} & \mathcal{T}[A-C](X) \\ \downarrow t_{A+(f_2/f_1)}^{A-C-(f_1)} & & \downarrow t_{A+(f_2)}^{A-C} \\ \mathcal{T}[A+(f_2/f_1)](X) & \xrightarrow[\mu_{1/f_1}^{A+(f_2/f_1)}]{} & \mathcal{T}[A+(f_2)](X) \end{array}$$

since $A-C-(f_1) \leq A+(f_2/f_1)$. Then we have

$$\begin{aligned} \phi_A \circ t_A^{A-C-(f_1)} &= \mathrm{Res}_{f_2\omega} \circ t_{A+(f_2)}^{A-C} \circ \mu_{1/f_1}^{A-C-(f_1)} \\ &= \mathrm{Res}_{f_2\omega} \circ \mu_{1/f_1}^{A+(f_2/f_1)} \circ t_{A+(f_2/f_1)}^{A-C-(f_1)} \\ &= \mathrm{Res}_{(f_2/f_1)\omega} \circ t_{A+(f_2/f_1)}^{A-C-(f_1)} \end{aligned}$$

Note that

$$((f_2/f_1)\omega) = (f_2/f_1) + (\omega) = (f_2) - (f_1) + (\omega) \geq A - C - (f_1).$$

So $(f_2/f_1)\omega \in L^{(1)}(A-C-(f_1))$. Then $\mathrm{Res}_{(f_2/f_1)\omega} : \mathcal{T}[A+(f_2/f_1)](X) \longrightarrow \mathbb{C}$ can be considered as a map $\mathrm{Res}_{(f_2/f_1)\omega} : \mathcal{T}[A-C-(f_1)](X) \longrightarrow \mathbb{C}$. Now if $Z \in \mathrm{Ker}(t_A^{A-C-(f_1)})$, then $\mathrm{Res}_{(f_2/f_1)\omega}(Z) = 0$. By Lemma 2.3.2, we have $(f_2/f_1)\omega$ belongs to $L^{(1)}(-A)$.

It follows we can consider the map $\mathrm{Res}_{(f_2/f_1)\omega}$ defined on $\mathcal{T}[A](X)$. Note that the following square commutes:

$$\begin{array}{ccc}
\mathcal{T}[A-C-(f_1)](X) & \xrightarrow{t^{A-C-(f_1)}_{A+(f_2/f_1)}} & \mathcal{T}[A+(f_2/f_1)](X) \\
{\scriptstyle t^{A-C-(f_1)}_{A}} \downarrow & & \downarrow {\scriptstyle \mathrm{Res}_{(f_2/f_1)\omega}} \\
\mathcal{T}[A](X) & \xrightarrow[\mathrm{Res}_{(f_2/f_1)\omega}]{} & \mathbb{C}
\end{array}$$

Then we have $\phi_A \circ t^{A-C-(f_1)}_A = \mathrm{Res}_{(f_2/f_1)\omega} \circ t^{A-C-(f_1)}_A$. Since $t^{A-C-(f_1)}_A$ is surjective, we get

$$\phi \circ t^A_D = \phi_A = \mathrm{Res}_{(f_2/f_1)\omega}.$$

Similarly, one can show that $(f_2/f_1)\omega \in L^{(1)}(-D)$ using Lemma 2.3.2. So $\mathrm{Res}_{(f_2/f_1)\omega} = \mathrm{Res}_{(f_2/f_1)\omega} \circ t^A_D$, and we get

$$\phi \circ t^A_D = \mathrm{Res}_{(f_2/f_!)\omega} \circ t^A_D.$$

Hence $\phi = \mathrm{Res}((f_2/f_1)\omega)$ since t^A_D is surjective.

□

Fix a canonical divisor $K = (\omega)$ for some meromorphic 1-form ω on X. By Proposition 1.1.3, we know that $L^{(1)}(-D)$ and $L(K-D)$ are isomorphic. Hence, by Serre Duality we get

$$\dim(H^1(D)) = \dim(L^{(1)}(-D)) = \dim(L(K-D)).$$

Finally, we show that $\dim(H^1(0)) = g$. Let K' be a canonical divisor on X of degree $2g-2$, which exists by Proposition 1.1.2. By Serre Duality, we have

$$\begin{aligned}
\dim(H^1(0)) &= \dim(L(K'-0)) = \dim(L(K')), \\
\dim(H^1(K')) &= \dim(L(K'-K')) = \dim(L(0)) = 1.
\end{aligned}$$

By the First Form of the Riemann-Roch Theorem and the previous equality, we have

$$\begin{aligned}
\dim(L(K')) - \dim(H^1(K')) &= \deg(K') + 1 - \dim(H^1(0)) \\
\dim(H^1(0)) - 1 &= 2g - 2 + 1 - \dim(H^1(0)) \\
\dim(H^1(0)) &= g.
\end{aligned}$$

We have obtained a more refined form of Theorem 2.2.1:

Theorem 2.3.2 (Riemann-Roch)**.** Let X be an algebraic curve of genus g. Then for any divisor D and any canonical divisor K, we have

$$\dim(L(D)) - \dim(L(K - D)) = \deg(D) + 1 - g.$$

2.4 Generalizations of the Riemann-Roch Theorem

In this section we shall comment three of the generalizations of the Riemann-Roch Theorem. We shall neither prove these results nor explain in detail the constructions used. We refer the reader to [1] and [2] for a more detailed exposition of this matter.

The first generalization that we shall give of the Riemann-Roch Theorem is its version for holomorphic line bundles. Let L be a holomorphic line bundle on a compact Riemann surface X of genus g and let $\Gamma(X, L)$ denote the space of holomorphic sections of L. This space is finite-dimensional. Let K denote the canonical bundle on X, i.e. $K = \Lambda^n(T^\vee)$ where $T^\vee$ is the cotangent bundle and $n = \dim(X)$.

Now consider a general section $\sigma : X \longrightarrow L$. We can produce such a section by giving it locally and then glueing it together using a partition of unity. Locally, the line bundle L is trivial so it looks like $\mathbb{C} \times \Delta \longrightarrow \Delta$ where Δ is the open unit disk. In this local picture, σ is just a map $\Delta \longrightarrow \mathbb{C}$. Locally, the inverse image of $0 \in \mathbb{C}$ under the map $\sigma : \Delta \longrightarrow \mathbb{C}$ has a finite number of points. Each point $p \in X$ where σ intersects the zero section is called a zero of σ. Around each such point p the section σ is a map $\sigma : \Delta \longrightarrow \mathbb{C}$ where $p = 0 \in \Delta$ and $\sigma(0) = 0$. The differential $T_p\sigma : T_0\Delta \longrightarrow T_0\mathbb{C}$ is a nonsingular two-by-two matrix. Let $\operatorname{sgn}(p)$ denote the sign of the determinant of this matrix. The degree of L is defined by $\deg(L) = \sum_{p \in X} \operatorname{sgn}(p)$ where the sum is over all points p where a section σ is zero.

The Riemann-Roch Theorem for holomorphic line bundles goes as follows:

Theorem 2.4.1. If L is a holomorphic line bundle on X and K is the canonical bundle on X, then

$$\dim(\Gamma(X, L)) - \dim(\Gamma(X, L^{-1} \otimes K)) = \deg(L) + 1 - g.$$

The Hirzebruch-Riemann-Roch Theorem is a result that contributes to the Riemann-Roch problem for complex algebraic varieties of all dimensions. The theorem applies to any holomorphic vector bundle E on a compact complex manifold X, to calculate the holomorphic Euler characteristic of E in sheaf cohomology, namely the alternating sum

$$\chi(X, E) = \dim(H^0(X, E)) - \dim(H^1(X, E)) + \dim(H^2(X, E)) - \cdots$$

of the dimensions as complex vector spaces.

Hirzebruch's theorem states that $\chi(X, E)$ is computable in terms of the Chern classes $C_j(E)$ of E, and the Todd polynomials T_j in the Chern classes of the holomorphic tangent bundle of X. These all lie in the cohomology ring of X.

Theorem 2.4.2. Let E be a holomorphic vector bundle on a compact complex manifold X. Then

$$\chi(X, E) = \sum_{j=0}^{n} \mathrm{ch}_{n-j}(E) \frac{T_j}{j!}$$

using the Chern character $\mathrm{ch}(E)$ in cohomology.

Finally, we comment the Grothendieck-Riemann-Roch Theorem. It is a generalization of the Hirzebruch-Riemann-Roch Theorem, about complex manifolds. Let X be a smooth quasi-projective scheme over a field. The Grothendieck group $K_0(X)$ of bounded complexes of coherent sheaves is canonically isomorphic to the Grothendieck group of bounded complexes of finite-rank vector bundles. Using this isomorphism, consider the Chern character as a functorial transformation

$$\mathrm{ch} : K_0(X) \longrightarrow A_d(X, \mathbb{Q})$$

where $A_d(X, \mathbb{Q})$ is the Chow group of cycles on X of dimension d modulo rational equivalence, tensored with the rational numbers. Now consider a proper morphism $f : X \longrightarrow Y$ between smooth quasi-projective schemes and a bounded complex of sheaves $\mathcal{F}^\bullet$. Let $\mathrm{td}(X)$ be the Todd genus of the tangent bundle of X. We denote the i-right derived functor of the pushforward f_* by $R^i f_*$. The Grothendieck-Riemann-Roch Theorem goes as follows:

Theorem 2.4.3. $\mathrm{ch}(f_! \mathcal{F}^\bullet)\mathrm{td}(Y) = f_*(\mathrm{ch}(\mathcal{F}^\bullet)\mathrm{td}(X))$, where

$$f_! : \sum (-1)^i R^i f_* K_0(X) \longrightarrow K_0(Y) \quad \text{and} \quad f_* : A(X) \longrightarrow A(Y).$$

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