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THE RADON-NIKODYM THEOREM

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Contents

Introduction

One of the most common problems studied in mathematical analysis is to find convenient representations of some special functions. For example, under certain conditions one can represent a linear functional on a Hilbert space in terms of the given scalar product. This fact is known as the Riesz Representation Theorem. A similar situation occurs in measure theory. Given two measures ν and μ on a measurable space $(X, \mathcal{M})$, a natural question that comes to us is if one can represent ν in terms of μ via some linear operator. The Radon-Nikodym theorem states that it is possible, under some hypothesis, to find a representation via the integral operator.

Specifically, this theorem states that, given a measurable space $(X, \mathcal{M})$, if a σ-finite measure ν on $(X, \mathcal{M})$ is absolutely continuous with respect to a σ-finite measure μ on $(X, \mathcal{M})$, then there is a non-negative measurable function f on X such that

$$\nu(E) = \int_E f d\mu,$$

for any measurable set E.

The theorem is named after Johann Radon, an austrian mathematician who proved the theorem for the special case where the underlying space is $\mathbb{R}^n$ in 1913, and for the polish mathematician Otton Nikodym who proved the general case in 1930.

The theorem is very important in extending the ideas of probability theory from probability masses and probability densities defined over real numbers to probability measures defined over arbitrary sets. It tells if and how it is possible to change from one probability measure to another. Specifically, the probability density function of a random variable is the Radon-Nikodym derivative of the induced measure with respect to some base measure (usually the Lebesgue measure for continuous random variables).

In the first chapter we introduce some notions about abstract measure theorem, such as signed measures. This notion gives rise to special kinds of sets called positive and negative sets. Then we prove the Hahn Decomposition Theorem, which states that given a signed measure ν on a measurable space X, it is possible to write X as a disjoint union of positive and negative sets. This theorem helps us to find a particular representation for ν as a difference of two non-negative measures, called the Jordan decomposition of ν.

In the second chapter we prove a version of the Radon-Nikodym theorem for the particular case where ν and μ are finite measures. Then we show an extension to σ-finite measures.

Finally, in the last chapter we use Jordan decompositions to extend the Radon-Nikodym Theorem to the case where ν is a σ-finite signed measure.

Chapter 1

Signed measures and decomposition theorems

In abstract measure theory, measures are sometimes allowed to take on both positive and negative values. Note that if μ_1 and μ_2 are two measures defined on a measurable space $(X, \mathcal{M})$ (i.e., X is a set and $\mathcal{M}$ is a σ-algebra of subsets of X), then we may define a new measure μ_3 on $(X, \mathcal{M})$ by setting

$$\mu_3(E) = c_1\mu_1(E) + c_2\mu_2(E),$$

where $c_1, c_2 \geq 0$. Some situations may arise if $c_1 < 0$ or $c_2 < 0$. For instance, what happen if $\mu_3(E) = \mu_1(E) - \mu_2(E)$. The first thing that may occur is that μ_3 is not always non-negative, and this leads to the consideration of signed measures. Another thing that may occur is that $\mu_1(E)$ and $\mu_2(E)$ are both $+\infty$. Thus $\mu_3(E)$ is not well defined. With these situations in mind, one seeks to define a function which satisfies the same conditions as any measure, and that is allowed to take negative values.

Definition 1. A **signed measure** on the measurable space $(X, \mathcal{M})$ is an extended real-valued set function ν defined for the sets of $\mathcal{M}$ and satisfying the following conditions:

(i) ν assumes at most one of the values $+\infty$, $-\infty$.

(ii) $\nu(\emptyset) = 0$.

(iii) $\nu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)$ for any sequence E_i of disjoint measurable sets, the equality taken to mean that the series on the right converges absolutely if $\nu(\cup_{i=1}^{\infty} E_i)$ is finite and that it properly diverges otherwise.

We shall say that a set A is a **positive set** with respect to a signed measure ν is A is measurable and for every measurable subset E of A we have $\nu(E) \geq 0$. Note that every measurable subset of a positive set is again positive, and if we take the restriction of ν to a positive set we obtain a positive measure. Similarly, B is called **negative** if it is a measurable set and for every measurable subset E of B we have $\nu(E) \leq 0$.

Example 1.

(1) Every measure is a signed measure, but a signed measure is not in general a measure.

(2) If $f \in L^1(\mu)$ then $\nu(E) = \int_E f d\mu$ is a signed measure.

Signed measures allow us to decompose the space X into the union of two disjoint measurable sets, one of them positive and the other one negative. This fact is known as the Hahn Decomposition Theorem. Before giving the formal statement and its proof, we prove two lemmas.

Lemma 1.0.1. Every measurable subset of a positive set is itself positive. The union of a countable collection of positive sets is positive.

Proof: The first statement is true by the definition of a positive set. Now we prove the second statement. Let A be the union of a sequence $\{A_n\}$ of positive sets. If E is any measurable subset of A, set $E_n = E \cap \left(A_n - \cup_{i=1}^{n-1} A_i\right)$. Each E_n is a measurable subset of A_n and so $\nu(E_n) \geq 0$. Since the E_n's are disjoint and $E = \cup_{n=1}^{\infty} E_n$, we have by the definition of a signed measure that $\nu(E) = \sum_{n=1}^{\infty} \nu(E_n) \geq 0$. Thus A is a positive set. $\square$

Lemma 1.0.2. Let E be a measurable set such that $0 < \nu(E) < \infty$. Then there is a positive set A contained in E with $\nu(A) > 0$.

Proof: If E is a positive set then there is nothing to prove. Suppose E is not a positive set. Then it contains sets of non-positive measure. Let n_1 be the smallest positive integer such that there is a measurable set $E_1 \subseteq E$ with $\nu(E_1) < -\frac{1}{n_1}$. Proceeding inductively, if $E - \cup_{j=1}^{k-1} E_j$ is not already a positive set, let n_k be the smallest positive integer for which there is a measurable set E_k such that $E_k \subseteq E - \cup_{j=1}^{k-1} E_j$ and $\nu(E_k) < -\frac{1}{n_k}$. Set $A = E - \cup_{k=1}^{\infty} E_k$. We shall see A is a positive set. Note that $E = A \cup (\cup_{k=1}^{\infty} E_k)$ and that this union is disjoint. Then we have $\nu(E) = \nu(A) + \nu\left(\cup_{k=1}^{\infty} E_k\right)$. On the other hand, $\sum_{k=1}^{\infty} |\nu(E_k)| = |\nu\left(\cup_{k=1}^{\infty} E_k\right)| \leq |\nu(E)| < \infty$ since every E_k is a subset of E. Thus $\sum_{k=1}^{\infty} \nu(E_k)$ converges absolutely and so we get $\nu(E) = \nu(A) + \sum_{k=1}^{\infty} \nu(E_k)$. Since $|\nu(E_k)| = -\nu(E_k) \geq \frac{1}{n_k}$ and $\sum_{k=1}^{\infty} |\nu(E_k)|$ converges, we have that $\sum_{k=1}^{\infty} \frac{1}{n_k}$ converges. Hence $\frac{1}{n_k} \longrightarrow 0$ as $k \longrightarrow \infty$, i.e., $\lim_{k\to\infty} n_k = +\infty$. Note that $\nu(E_k) \leq 0$ and $\nu(E) > 0$ implies that $\nu(A) > 0$. Now let $\epsilon > 0$ be given. Since $n_k \longrightarrow \infty$, we may choose k so large that $(n_k - 1)^{-1} < \epsilon$. Suppose A contains a set with measure less than $-(n_k - 1)^{-1}$, say C, i.e., $C \subseteq A \subseteq E - \left(\cup_{j=1}^{k-1} E_j\right)$ and $\nu(C) < -\frac{1}{n_k - 1} < -\frac{1}{n_k}$. We get a contradiction since

n_k is the smallest positive integer for which there is a measurable set $E_k \subseteq E - \left(\cup_{j=1}^{k-1} E_j\right)$ such that $\nu(E_k) < -\frac{1}{n_k}$. Then A can contain no measurable sets with measure less than $-(n_k - 1)^{-1}$, which is greater than $-\epsilon$. Thus A contains no measurable sets of measure less that $-\epsilon$. Since ϵ is an arbitrary positive number, it follows that A can contain no sets of negative measure and so must be a positive set. □

Now we are ready to prove the Hahn Decomposition Theorem.

Proposition 1.0.1 (Hahn Decomposition Theorem). Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \emptyset$.

Proof: Since ν assumes at most one of the values $-\infty$ and $+\infty$, we may suppose that $-\infty \leq \nu(E) < +\infty$ for every measurable set E. Let λ be the supremum $\sup\{\nu(A) : A \text{ is a positive set with respect to } \nu\}$. Since $\emptyset$ is positive and $\nu(\emptyset) = 0$, we have that $\lambda \geq 0$. Let $\{A_i\}$ be a sequence of positive measurable sets such that $\lambda = \lim_{i\to\infty} \nu(A_i)$, and set $A = \cup_{i=1}^{\infty} A_i$. By Lemma 1.0.1 the set A is itself a positive set, and so $\nu(A - A_i) \geq 0$. Thus $\nu(A) = \nu(A_i) + \nu(A - A_i) \geq \nu(A_i)$, for every i. Hence $\lambda = \lim_{i\to\infty} \nu(A_i) \leq \nu(A) \leq \lambda$. We get $\nu(A) = \lambda$ and $\lambda < \infty$. Let $B = X - A$. We shall see that B is a negative set. Suppose that E is a positive subset of B. Then E and A are disjoint and $E \cup A$ is a positive set. Hence $\lambda \geq \nu(E \cup A) = \nu(E) + \nu(A) = \nu(E) + \lambda$. Since $\lambda < \infty$, we have $\nu(E) = 0$. Thus B contains no positive subsets of positive measure. Now if B contains a subset of positive measure then by Lemma 1.0.2 there exists a positive set $A \subseteq E$ such that $\nu(A) > 0$. Since B contains no positive subsets of positive measure, we have that B contains no subsets of positive measure, i.e., B is a negative set. □

Definition 2. If $(X, \mathcal{M})$ is a measurable space and μ and ν are signed measures on $\mathcal{M}$, we say that μ and ν are **mutually singular**, in symbols $\mu \perp \nu$, if there exist two disjoint sets A and B whose union is X such that, for every measurable set E, $|\mu|(A \cap E) = |\nu|(E \cap B) = 0$.

Proposition 1.0.2 (Jordan Decomposition Theorem). Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{M})$ such that $\nu = \nu^+ - \nu^-$. Moreover, such a pair (ν^+, ν^-) is unique.

Proof: Let $\{A, B\}$ be a Hahn decomposition for ν, then define $\nu^+ : \mathcal{M} \longrightarrow [0, +\infty]$ and $\nu^- : \mathcal{M} \longrightarrow [0, +\infty]$ by setting $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$. Since A (resp. B) is a positive (resp. negative) set with respect to ν, and $E \cap A \subseteq A$ (resp. $E \cap B \subseteq B$), we have ν^+ and ν^- assume non-negative values. Clearly ν^+ and ν^- are measures. Now we show that ν^+ and ν^- are mutually singular. Since A and B are disjoint measurable sets such that $X = A \cup B$ and

$$\begin{aligned}\nu^+(B \cap E) &= \nu(E \cap (B \cap A)) = \nu(\emptyset) = 0,\\ \nu^-(A \cap E) &= -\nu((A \cap B) \cap E) = -\nu(\emptyset) = 0,\end{aligned}$$

for every measurable set E, we have that ν^+ and ν^- are mutually singular. It is only left to show that $\nu = \nu^+ - \nu^-$. We have

$$\begin{aligned}\nu(E) &= \nu(E \cap X) = \nu(E \cap (A \cup B))\\ &= \nu((E \cap A) \cup (E \cap B)) = \nu(E \cap A) + \nu(E \cap B),\\ &\quad \text{since } E \cap A \text{ and } E \cap B \text{ are disjoint, and } \nu \text{ is a signed measure}\\ &= \nu^+(E) - \nu^-(E).\end{aligned}$$

Now we shall prove that the decomposition $\nu = \nu^+ - \nu^-$ is unique, in the sense that ν^+ and ν^- does not depend on the chosen Hahn decomposition. Let $X = A' \cup B'$ be another Hahn decomposition for ν. We prove that, for every measurable set E, $\nu(E \cap A) = \nu(E \cap A')$ and $\nu(E \cap B) = \nu(E \cap B')$. Observe that $E \cap (A - A') \subseteq A$, so that $\nu(E \cap (A - A')) \geq 0$, and $E \cap (A - A') \subseteq B'$, so that $\nu(E \cap (A - A')) \leq 0$. Hence $\nu(E \cap (A - A')) = 0$ and, by symmetry, $\nu(E \cap (A' - A)) = 0$. Notice that $A \cup A' = A \cup (A' - A)$, then

$$\begin{aligned}\nu(E \cap (A \cup A')) &= \nu(E \cap (A \cup (A' - A))) = \nu((E \cap A) \cup (E \cap (A' - A)))\\ &= \nu(E \cap A) + \nu(E \cap (A' - A)) = \nu(E \cap A) + 0\\ &= \nu(E \cap A).\end{aligned}$$

Similarly, $\nu(E \cap (A \cup A')) = \nu(E \cap A')$. In a similar way, one can prove that $\nu(E \cap B) = \nu(E \cap B')$. It follows from this result that $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$ unambiguously define two functions ν^+ and ν^- on the class of all measurable sets. $\square$

Definition 3. If $(X, \mathcal{M})$ is a measurable space and μ and ν are signed measures on $\mathcal{M}$, we say that ν is **absolutely continuous** with respect to μ, in symbols $\nu \ll \mu$, if $\nu(E) = 0$ for every measurable set E for which $|\mu|(E) = 0$.

Remark 1. Given a signed measure ν and (ν^+, ν^-) its Jordan decomposition. We denote $|\nu| = \nu^+ + \nu^-$.

Proposition 1.0.3. If ν and μ are signed measures, then the conditions

(a) $\nu \ll \mu$,

(b) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$,

(c) $|\nu| \ll |\mu|$,

are mutually equivalent.

Proof:

- (a)$\Longrightarrow$ (b): Let E be a measurable set such that $|\mu|(E) = 0$. Then

$$0 \le |\mu|(A \cap E) \le |\mu|(E) = 0,$$

  i.e., $|\mu|(A \cap E) = 0$. Similarly, $|\mu|(B \cap E) = 0$. Since $\nu \ll \mu$, we have that $\nu(A \cap E) = 0$ and $\nu(B \cap E) = 0$, i.e., $\nu^+(E) = \nu^-(E) = 0$. Hence $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

- (b)$\Longrightarrow$ (c): Let E be a measurable set such that $|\mu|(E) = 0$. Then

$$\begin{aligned} \nu^+ \ll \mu &\implies \nu^+(E) = 0, \\ \nu^- \ll \mu &\implies \nu^-(E) = 0. \end{aligned}$$

  Thus $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0 + 0 = 0$. Hence $|\nu| \ll |\mu|$.

- (c)$\Longrightarrow$ (a): Let E be a measurable set such that $|\mu|(E) = 0$. Since $|\nu| \ll |\mu|$ we have $|\nu|(E) = 0$. Thus $0 \le \nu^+(E) \le |\nu|(E) = 0$, and hence $\nu^+(E) = 0$. Similarly $\nu^-(E) = 0$. Then $\nu(E) = \nu^+(E) - \nu^-(E) = 0 - 0 = 0$. Therefore $\nu \ll \mu$.

$\square$

Chapter 2

The Radon-Nikodym Theorem

Lemma 2.0.3. If μ and ν are finite measures such that $\nu \ll \mu$ and ν is not identically zero, then there exist a positive number ϵ and a measurable set A such that $\mu(A) > 0$ and such that A is a positive set for the signed measure $\nu - \epsilon\mu$.

Proof: For each $n = 1, 2, \ldots$, consider the signed measure $\nu - \frac{1}{n}\mu$ and let $X = A_n \cup B_n$ be a Hahn decomposition with respect to $\nu - \frac{1}{n}\mu$. We write $A_0 = \cup_{n=1}^{\infty} A_n$, $B_0 = \cap_{n=1}^{\infty} B_n$. Since $B_0 \subseteq B_n$ for each $n \in \mathbb{Z}_+$ and each B_n is a negative set, we have $0 \leq \nu(B_0) \leq \frac{1}{n}\mu(B_0)$, for every $n \in \mathbb{Z}_+$. Hence $\nu(B_0) = 0$. Notice that $X = A_0 \cup B_0$ is a disjoint union. If $\nu(A_0) = 0$ then $\nu(X) = \nu(A_0) + \nu(B_0) = 0 + 0 = 0$. We get $\nu(X) = 0$. However, this is not possible since ν is not identically zero. Thus $\nu(A_0) > 0$. Since $\nu \ll \mu$ we have $\mu(A_0) > 0$. On the other hand, $0 < \mu(A_0) = \mu\left(\cup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$, since μ is σ-subadditive. Hence we must have $\mu(A_N) > 0$ for at least one value of N. We set $A = A_N$ and $\epsilon = 1/N$. Therefore we have that $\mu(A) > 0$ and $\nu(A) - \epsilon\mu(A) > 0$. $\square$

Theorem 2.0.1. If $(X, \mathcal{M}, \mu)$ is a measure space, where μ is a finite measure, and if ν is a finite measure on $\mathcal{M}$ absolutely continuous with respect to μ, then there exists a finite valued measurable function f on X such that

$$\nu(E) = \int_E f d\mu, \quad (*)$$

for every measurable set E. The function f is unique in the sense that if g satisfies $(*)$ for every $E \in \mathcal{M}$, then $f = g$ a.e.(μ).

Proof: Let $\mathcal{C}$ be the class of all non-negative functions f, integrable with respect to μ, such that $\int_E f d\mu \le \nu(E)$ for every measurable set E, and write $\alpha = \sup\left\{\int f d\mu : f \in \mathcal{C}\right\}$. Notice that $\mathcal{C} \neq \emptyset$ since $0 \in \mathcal{C}$. Moreover,

$$0 \le \int f d\mu \le \nu(X) < \infty, \text{ for every } f \in \mathcal{C} \Longrightarrow 0 \le \alpha < \infty.$$

Since α is the limit point of the set $\{\int f d\mu : f \in \mathcal{C}\}$, there exists a sequence of functions in $\mathcal{C}$ such that $\alpha = \lim_{n\to\infty} \int f_n d\mu$. Let E a measurable set and n a positive integer. Define a function $g_n : X \longrightarrow [0, +\infty]$ by $g_n = \max\{f_1, \ldots, f_n\}$. We have that g_n is a non-negative measurable function since it is the maximum of non-negative measurable functions. Let

$$A_i = E \cap \left(\bigcap_{k=1, k\neq i}^{n} (f_i - f_k)^{-1}([0, +\infty)) \right),$$

for $i = 1, \ldots, n$. Now set $E_1 = A_1$, $E_2 = A_2 - A_1$, $\ldots$, $E_n = A_n - (\cup_{i=1}^{n-1} A_i)$. Thus, $E = E_1 \cup \cdots \cup E_n$ is a disjoint union such that $g_n(x) = f_i(x)$ for $x \in E_i$. Note that g_n is an integrable function since it can be written as $g_n = \sum_{i=1}^n f_i \chi_{E_i}$. Consequently we have

$$\begin{aligned}\int_E g_n d\mu &= \int_E \left(\sum_{i=1}^n f_i \chi_{E_i}\right) d\mu = \sum_{i=1}^n \int_E f_i \chi_{E_i} d\mu \\ &= \sum_{i=1}^n \int_{E_i} f_i d\mu \le \sum_{i=1}^n \nu(E_i) \\ &= \nu(E) \text{ since } \nu \text{ is a measure and the } E_i's \text{ are disjoint.}\end{aligned}$$

Hence we have that $g_n \in \mathcal{C}$. Let $f_0 : X \longrightarrow [0, +\infty]$ be the function

$$f_0(x) = \sup\{f_n(x) : n = 1, 2, \ldots\}.$$

We have $f_0(x) = \lim_{n\to\infty} g_n(x)$. Then, $\{g_n\}$ is a non-decreasing sequence of non-negative measurable functions that converges point-wise to f_0. By the Monotone Convergence Theorem we have $\int_X f_0 d\mu = \lim_{n\to\infty} \int_X g_n d\mu$. On the other hand, $\int_X g_n d\mu$ is a non-decreasing sequence in $\{\int_X f d\mu : f \in \mathcal{C}\}$ since each $g_n \in \mathcal{C}$. Then $\int_X g_n d\mu \le \alpha$ for every n, and hence

$$\int_X f_0 d\mu = \lim_{n\to\infty} \int_X g_n d\mu \le \alpha. \quad (1)$$

Also, since $f_n \le g_n$ for every n, we have $\int_X f_n d\mu \le \int_X g_n d\mu$ for every n. Then

$$\alpha = \lim_{n\to\infty} \int_X f_n d\mu \le \lim_{n\to\infty} \int_X g_n d\mu = \int_X f_0 d\mu. \quad (2)$$

From (1) and (2) we have $\int_X f_0 d\mu = \alpha$. Moreover,

$$\int_E f_0 d\mu = \lim_{n\to\infty} \int_E g_n d\mu \le \lim_{n\to\infty} \nu(E) = \nu(E).$$

Hence $f_0 \in \mathcal{C}$. Since f_0 is an integrable function, there exists a finite valued function f such that $f = f_0$ a.e.(μ). Let $\nu_0 : \mathcal{M} \longrightarrow [0, +\infty]$ be the function given by $\nu_0(E) = \nu(E) - \int_E f d\mu$. In fact, $\nu_0(E) \geq 0$ since $\int_E f d\mu = \int_E f_0 d\mu \leq \nu(E)$. We have that ν_0 is a measure. Also ν_0 is finite since ν is and f is integrable on each E. Note that $\nu_0 \ll \mu$. For if $\mu(E) = 0$ then $\nu(E) = 0$ since $\nu \ll \mu$, and $\int_E f d\mu = 0$; so $\nu_0(E) = 0$. We prove that ν_0 is identically zero. Suppose the converse. Then ν_0 satisfies the hypothesis of the previous lemma, so there exists $\epsilon > 0$ and a measurable set A such that $\mu(A) > 0$ and such that A is a positive set for the signed measure $\nu_0 - \epsilon\mu$. Let E be a measurable set, then $E \cap A \subseteq A$ is a measurable set, and since A is positive for $\nu_0 - \epsilon\mu$ we have $\nu_0(E \cap A) - \epsilon\mu(E \cap A) \geq 0$, i.e. $\epsilon\mu(E \cap A) \leq \nu_0(E \cap A) = \nu(E \cap A) - \int_{E \cap A} f d\mu$. If $g = f + \epsilon\chi_A$ then g is integrable and

$$\begin{aligned}
\int_E g d\mu &= \int_E f d\mu + \int_E \epsilon\chi_A d\mu \quad \text{by linearity} \\
&= \int_E f d\mu + \epsilon\mu(E \cap A) \\
&= \int_{E \cap A} f d\mu + \int_{E - A} f d\mu + \epsilon\mu(E \cap A) \quad \text{since the integral is additive} \\
&\leq \int_{E \cap A} f d\mu + \int_{E - A} f d\mu + \nu(E \cap A) - \int_{E \cap A} f d\mu \\
&= \int_{E - A} f d\mu + \nu(E \cap A) = \int_{E - A} f_0 d\mu + \nu(E \cap A) \\
&\leq \nu(E - A) + \nu(E \cap A), \quad \text{since } f_0 \in \mathcal{C} \\
&= \nu(E) \quad \text{since } \nu \text{ is additive.}
\end{aligned}$$

Hence $g \in \mathcal{C}$. However, $\int_X g d\mu = \int_X f d\mu + \epsilon\mu(A) > \alpha$, since $\epsilon\mu(A) > 0$ and $\int_X f d\mu = \int_X f_0 d\mu = \alpha$. We get a contradiction since α is the supremum of the set $\{\int_X f d\mu : f \in \mathcal{C}\}$ and $\int_X g d\mu \in \{\int_X f d\mu : f \in \mathcal{C}\}$. Hence $\nu_0(E) = 0$ for every measurable set E, i.e., $\nu(E) = \int_E f d\mu$.

Now we prove uniqueness of the function f. Let g be another non-negative measurable function satisfying $(*)$. Since $\nu(E) < \infty$ for every $E \in \mathcal{M}$, we have that g is integrable on every $E \in \mathcal{M}$. By linearity of the Lebesgue integral for integrable functions, we have

$$0 = \nu(E) - \nu(E) = \int_E g d\mu - \int_E f d\mu = \int_E (g - f) d\mu,$$

for every $E \in \mathcal{M}$. Then we have $f = g$ a.e.(μ). □

Now we give an extension of the previous theorem to the case where μ and ν are σ-finite measures.

Theorem 2.0.2 (Radon-Nikodym Theorem for σ-finite measures)**.** Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let ν be a σ-finite measure such that ν is absolutely continuous with respect to μ. Then there exists a non-negative measurable function f such that

$$\nu(E) = \int_E f d\mu \qquad (*),$$

for every measurable set E. The function f is unique in the sense that if g is another non-negative measurable function satisfying $(*)$ for every measurable set E, then $g = f$ a.e.(μ).

Proof: Since μ and ν are σ-finite measures, there exist pairwise disjoint collections $\{X_i\}_{i=1}^{\infty}$ and $\{Y_j\}_{j=1}^{\infty}$ of sets in $\mathcal{M}$ such that $X = \bigcup_{i=1}^{\infty} X_i = \bigcup_{j=1}^{\infty} Y_j$, and $\mu(X_i), \nu(Y_j) < \infty$ for every $i, j = 1, \ldots, +\infty$. We have

$$X_i = X_i \cap X = X_i \cap \left(\bigcup_{j=1}^{\infty} Y_j \right) = \bigcup_{j=1}^{\infty} X_i \cap Y_j,$$

$$X = \bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{\infty} X_i \cap X_j \right) = \bigcup_{i,j} X_i \cap Y_j.$$

Let $E_{ij} = X_i \cap Y_j$. Note that $E_{ij} \cap E_{pq} = \emptyset$ if and only if $i \neq p$ or $j \neq q$. Then $\{E_{ij}\}_{i,j=1}^{\infty}$ is a pairwise disjoint collection of measurable sets whose union is X. Define a set function $\mu_{ij} : \mathcal{M} \longrightarrow [0, +\infty]$ by $\mu_{ij}(A) = \mu(A \cap E_{ij})$, for every $A \in \mathcal{M}$. Clearly, each μ_{ij} is a measure. Also, $\mu_{ij}(X) = \mu(X \cap E_{ij}) \leq \mu(X_i) < \infty$. Then each μ_{ij} is a finite measure. In a similar way, we define for each pair (i, j) a finite measure $\nu_{ij} : \mathcal{M} \longrightarrow [0, +\infty]$ by $\nu_{ij}(A) = \nu(A \cap E_{ij})$, for every measurable set A. We show that $\nu_{ij} \ll \mu_{ij}$. Let A be a measurable set such that $\mu_{ij}(A) = 0$. Then $\mu(A \cap E_{ij}) = 0$. Since $\nu \ll \mu$ we have that $\nu(A \cap E_{ij}) = 0$, i.e., $\nu_{ij}(A) = 0$. Then $\nu_{ij} \ll \mu_{ij}$. By the Radon-Nikodym theorem, for each pair (i, j) there exists a finite valued measurable function f_{ij} on X such that $\nu_{ij}(E) = \int_E f_{ij} d\mu_{ij}$, for every measurable set E. Note that $\nu_{ij}(X - E_{ij}) = \nu(E_{ij} - E_{ij}) = \nu(\emptyset) = 0$. Then $\int_{X - E_{ij}} f_{ij} d\mu_{ij} = 0$. Thus we can set $f_{ij}(x) = 0$ if $x \notin E_{ij}$. Let f be the function $f(x) = f_{ij}(x)$ if $x \in E_{ij}$. Note that f is well defined since the E_{ij}'s are disjoint. By the last comment in the previous paragraph, f can be written as $f = \sum_{i,j=1}^{\infty} f_{ij}$. Then f is a non-negative measurable function since it is the pointwise limit of non-negative measurable functions. Let $E \in \mathcal{M}$. We shall see that $\nu(E) = \int_E f d\mu$.

We have

$$
\begin{aligned}
\nu(E) &= \nu(E \cap X) = \nu(E \cap (\cup_{i,j} E_{ij})) && (2.1)\\
&= \nu(\cup_{i,j} E \cap E_{ij}) = \sum_{i,j} \nu(E \cap E_{ij}) && (2.2)\\
&= \sum_{i,j} \nu_{ij}(E) = \sum_{i,j} \int_E f_{ij} d\mu_{ij} && (2.3)\\
&= \sum_{i,j} \int_E f d\mu_{i,j} = \sum_{i,j} \int_{E\cap E_{ij}} f d\mu && (2.4)\\
&= \sum_{i,j} \int_E f \cdot \chi_{E_{ij}} d\mu = \int_E \left(\sum_{i,j} f \cdot \chi_{E_{ij}} \right) d\mu && (2.5)\\
&= \int_E f d\mu. && (2.6)
\end{aligned}
$$

Equality (2.2) holds since ν is a measure and the $E \cap E_{ij}$ are disjoint. Equality (2.5) is a corollary of the Monotone Convergence Theorem for series of functions pointwise convergent. The only equality which requires proof is (2.4).

Claim 1. For every measurable set E,

$$\int_E f d\mu_{ij} = \int_{E\cap E_{ij}} f d\mu.$$

Proof: Recall that

$$\int_E f d\mu_{ij} = \sup \left\{ \int_E \varphi d\mu_{ij} : 0 \le \varphi \le f \text{ and } \varphi \text{ is simple} \right\},$$

and

$$\int_{E\cap E_{ij}} f d\mu = \sup \left\{ \int_{E\cap E_{ij}} \psi d\mu : 0 \le \psi \le f \text{ and } \psi \text{ is simple} \right\}.$$

Thus we only have to prove the statement for simple functions. Let φ be a simple function. Then φ can be written as $\varphi = \sum_{k=1}^m \alpha_k \cdot \chi_{A_k}$, where each A_k is a measurable set. We have $\int_E \varphi d\mu_{ij} = \sum_{k=1}^m \alpha_k \cdot \mu_{ij}(A_k \cap E) = \sum_{k=1}^m \alpha_k \cdot \mu(A_k \cap E \cap E_{ij}) = \int_{E\cap E_{ij}} \varphi d\mu$. □

To complete the proof of the theorem, it is only left to show that the function f is unique in the "a.e.(μ)" sense. Suppose there is another non-negative measurable function g such that $\nu(E) = \int_E g d\mu$ for every measurable set E. Then

$$\int_E f d\mu_{ij} = \int_{E\cap E_{ij}} f d\mu = \nu(E \cap E_{ij}) = \int_{E\cap E_{ij}} g d\mu = \int_E g d\mu_{ij}.$$

By the uniqueness stated in the Radon-Nikodym Theorem, we have that $g = f$ a.e.(μ_{ij}). Now let $D = \{x \in X : f(x) \neq g(x)\}$. We have

$$\mu(D) = \mu(D \cap X) = \mu(D \cap (\cup_{i,j} E_{ij})) = \mu(\cup_{i,j} D \cap E_{ij}) = \sum_{i,j} \mu(D \cap E_{ij})$$
$$= \sum_{i,j} \mu_{ij}(D) = \sum_{i,j} 0 = 0.$$

Hence $g = f$ a.e.(μ). $\square$

Example 2. The previous theorem is not true in general if we drop the assumption that μ is σ-finite. Let $X = \mathbb{R}$, $\mathcal{M} = \{E \subseteq \mathbb{R} :$ either E or E^c is countable$\}$ and μ the counting measure. Define a measure ν by

$$\nu(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ 1 & \text{otherwise.} \end{cases}$$

Note that $\nu \ll \mu$ and μ is not σ-finite. Suppose there exists a non-negative measurable function f such that $\nu(E) = \int_E f d\mu$. If $f \equiv 0$ then $1 = \nu(\mathbb{R}) = \int_{\mathbb{R}} 0 d\mu = 0$. Thus assume that f is not identically zero. Then there exists $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$. Let $E = \{x_0\}$. Then $E \in \mathcal{M}$ and $\nu(E) = 0$. We have $0 = \nu(E) = \int_E f d\mu = f(x_0) > 0$. In both cases we get a contradiction.

Chapter 3

Extensions of the Radon-Nikodym Theorem to signed measures

In this chapter we shall see that the Radon-Nikodym Theorem also holds if ν is a finite (or σ-finite) signed measure. In fact, the idea is to apply the Radon-Nikodym Theorem to each pair of measures (ν^+, μ) and (ν^-, μ). We prove first the easiest case when ν is a finite signed measure, and then we extend this result to σ-finite signed measures.

Theorem 3.0.3 (First extension)**.** Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let ν be a finite signed measure such that $\nu \ll \mu$. Then there exists a measurable function f such that $\nu(E) = \int_E f d\mu$, for every measurable set E. If there exists another measurable function g satisfying the equality above, then we have that $g = f$ a.e.(μ).

Proof: Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. By Proposition 1.0.3 we have $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. Clearly ν^+ and ν^- are also finite measures. Then by the Radon-Nikodym Theorem we have that there exist non-negative measurable functions f_1 and f_2 such that $\nu^+(E) = \int_E f_1 d\mu$ and $\nu^-(E) = \int_E f_2 d\mu$, for every measurable set E. Since f_1 and f_2 are both integrable, we get

$$\nu(E) = \nu^+(E) - \nu^-(E) = \int_E f_1 d\mu - \int_E f_2 d\mu = \int_E (f_1 - f_2) d\mu.$$

Hence set $f = f_1 - f_2$. The uniqueness of f follows easily, it is the same argument we used in the proof of Theorem 2.0.1. □

Theorem 3.0.4 (Second extension)**.** Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let ν be a σ-finite signed measure such that $\nu \ll \mu$. Then there exists a measurable function f such that $\nu(E) = \int_E f d\mu$, for every measurable set E. If there exists another measurable function g satisfying the equality above, then we have that $g = f$ a.e.(μ).

Proof: Using the same argument from the previous theorem, we have that there exist non-negative-measurable functions f_1 and f_2 such that $\nu^+(E) = \int_E f_1 d\mu$ and $\nu^-(E) = \int_E f_2 d\mu$, for every measurable set E. These functions need not be integrable on E, then we may not apply linearity as in the previous proof. We have to study several cases.

(1) f_1 and f_2 are integrable on E: Then we have $\nu(E) = \int_E f d\mu$, where $f = f_1 - f_2$.

(2) Both $\int_E f_1 d\mu$ and $\int_E f_2$ are ∞: This case is not possible since ν assumes at most one of the values ∞ and $-\infty$. We would get $\nu(E) = \infty - \infty$.

(3) $\int_E f_1 d\mu = \infty$ and $\int_E f_2 d\mu < \infty$: We have $f^+ - f^- = f_1 - f_2$. Thus, $f^+ + f_2 = f^- + f_1$. Since the integral is linear for non-negative functions, we have

$$\int_E f^+ d\mu + \int_E f_2 d\mu = \int_E f^- d\mu + \int_E f_1 d\mu.$$

On the other hand, $\int_E f_2 d\mu < \infty$, then

$$\int_E f^+ d\mu = \int_E f^- d\mu + \int_E f_1 d\mu - \int_E f_2 d\mu = \int_E f^- d\mu + \int_E f_1 d\mu.$$

It follows $\int_E f^+ d\mu = \infty$. Since $\nu(E) = \infty$, we have to prove that $\int_E f^+ d\mu - \int_E f^- d\mu = \infty$. We shall see that $\int_E f^- d\mu < \infty$. Let $C = f^{-1}(-\infty, 0)$. Then

$$\int_{E \cap C} f^- d\mu = -\int_{E \cap C} f d\mu.$$

Also, we have that $0 \leq f_1 < f_2$ on $E \cap C$ and then $\int_{E\cap C} f_2 d\mu < \infty$ implies that $\int_{E \cap C} f_1 d\mu < \infty$. Hence

$$\begin{aligned}\int_E f^- d\mu &= \int_{E\cap C} f^- d\mu + \int_{E-C} f^- d\mu = -\int_{E\cap C} f d\mu + \int_{E-C} 0 d\mu \\ &= -\int_{E\cap C} (f_1 - f_2) d\mu = \int_{E\cap C} f_2 d\mu - \int_{E\cap C} f_1 d\mu \\ &< \infty.\end{aligned}$$

Finally, we have $\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = \infty = \nu(E)$.

(4) The case $\int_E f_2 d\mu = \infty$ and $\int_E f_1 d\mu < \infty$ is similar to (3).

Therefore, $\nu(E) = \int_E f d\mu$ for every measurable set E.

Now we prove that f is the only measurable function satisfying the equality above. Let g be another measurable function satisfying $\nu(E) = \int_E g d\mu$. Let $X = \{A, B\}$ be a Hahn decomposition of X. We have that $f_1|_B = 0$ since $\int_E f_1 d\mu = \nu^+(E) = \nu(A \cap E) = 0$ for every $E \subseteq B$. Similarly, $f_2|_A = 0$. On the other hand,

$$\int_E g d\mu = \nu(E) = \nu^+(E) = \int_E f_1 d\mu,$$

for every measurable set $E \subseteq A$. By the uniqueness of f_1, it follows that $g = f_1$ a.e.(μ) on A. Similarly, $g = f_2$ a.e.(μ) on B. Therefore $g = f_1 - f_2 = f$ a.e.(μ). □

Remark 2. This theorem has one more extension. In fact, there is no need to assume that ν is σ-finite. In these notes, however, one gave a proof that uses strongly that fact. One considered that this prove is more constructive and easier to follow. You can check [2] if you are interested in a proof where ν is not σ-finite.

Bibliography

[1] Halmos, P. *Measure Theory*. Springer-Verlag. New York. (1974).

[2] Royden, H. *Real Analysis*. Pearson Education, Inc. Singapore. (1988).