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THE MAXIMAL TORUS OF A COMPACT LIE GROUP

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Introduction

If G is a compact connected Lie group, we say that T is a torus if it is a connected abelian subgroup of G. A torus T is said to be maximal if given another torus T' such that $T \subseteq T'$, then $T = T'$. An equivalent definition of a torus is that of a Lie subgroup of G isomorphic to $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, for some $k \in \mathbb{N}$. It is well known every compact connected Lie group has a torus and that every torus in contained in a maximal torus. The fact that G is compact is of vital importance. A non compact Lie group need not have any nontrivial tori, for example $\mathbb{R}^n$.

Maximal tori in a compact connected Lie group G have some interesting properties. Among them, the most important are known as the conjugacy theorems.

Theorem (First Conjugacy Theorem)**.** In a compact connected Lie group G, every element is conjugate to an element in any fixed maximal torus.

Theorem (Second Conjugacy Theorem)**.** Any two maximal tori in a compact connected Lie group G are conjugate.

The purpose of these notes is to give a self contained proof of the conjugacy theorems. In the first chapter we recall some basic notions on Lie group theory, such as the exponential map and integration. We also study the concept of mapping degree of a smooth map, which shall play an important role in the proof of the conjugacy theorems.

We start the second chapter studying some notions related to maximal tori, such as Weyl groups. We prove that the Weyl group associated to a torus is finite. As a consequence of the Second Conjugacy Theorem, we prove that any two Weyl groups are isomorphic. Then we give the proof for both First and Second Conjugacy Theorems, which are based on [2]. Finally, we show some consequences of these theorems. For example, the exponential map of a compact connected Lie group is surjective. Other consequences are related to the notion of centralizer. We prove that the centralizer of any maximal torus is the torus itself.

Chapter 1

Lie Groups and Lie Algebras

1.1 Some basic notions on Lie groups

Recall that a **Lie Group** G is a differentiable manifold with a group structure given by maps:

$$\begin{aligned} m &: G \times G \longrightarrow G \quad (g,h) \mapsto gh \quad \text{(multiplication)}, \\ i &: G \longrightarrow G \qquad\quad g \mapsto g^{-1} \qquad \text{(inversion)}, \end{aligned}$$

such that m and i are smooth maps. A **Lie algebra** is a finite dimensional vector space $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$) with a multiplication $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, called the **Lie bracket** such that:

(1) $[X,Y] = -[Y,X]$,

(2) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

A vector field $X : G \longrightarrow TG$ is called **left-invariant** if for every $g, h \in G$ we have

$$(dl_g)(X_h) = X_{l_g(h)},$$

where $l_g : G \longrightarrow G$ is the left translation $h \mapsto gh$ and $(dl_g) : T_hG \longrightarrow T_{l_g(h)}G$ is the differential map. We shall denote $\mathfrak{g}$ the set of all left-invariant vector fields on a Lie group G with the Lie bracket $[\cdot,\cdot]$ for vector fields, i.e.,

$$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f)),$$

where X and Y are vector fiends on G, $p \in G$, and f is a smooth function on a neighbourhood of p.

Proposition 1.1.1.

(1) $\mathfrak{g}$ is isomorphic to $T_e(G)$, where e denotes the identity element of G.

(2) $[X, Y] \in \mathfrak{g}$, foe every $X, Y \in \mathfrak{g}$.

(3) $\mathfrak{g}$ is a Lie algebra

Proof: See [4, Example 4.19 and Theorem 4.20]. □

A map $F : G \longrightarrow H$ is a **homomorphism of Lie groups** if it is a smooth group homomorphism. A map $\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a **homomorphism of Lie algebras** if it is linear and satisfies the equality

$$\psi([X, Y]) = [\psi(X), \psi(Y)].$$

A **Lie subgroup** H of G is a Lie group which is a subgroup of G and the inclusion $i : H \longrightarrow G$ is an injective immersion. A **closed Lie subgroup** of G is a Lie subgroup H such that $i(H)$ is closed in G.

Theorem 1.1.1 (Closed Subgroup Theorem)**.** If H is a subgroup of a Lie group G that is also a closed subset of G, then H is an embedded Lie subgroup.

Proof: See [4, Theorem 20.10]. □

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $V \in \mathfrak{g}$ and consider the local flow generated by V, say φ. It is well known that every left-invariant vector field has a global flow (see [2, Chapter 1]). Then there exists a Lie group homomorphism $\varphi : \mathbb{R} \longrightarrow G$ such that $d\varphi(1) = V$. The **exponential map** is the application $\exp : \mathfrak{g} \longrightarrow G$ given by $V \longrightarrow \varphi(1)$.

Proposition 1.1.2 (Properties of exp)**.** Let $F : H \longrightarrow G$ be a homomorphism of Lie groups. Then the following diagram is commutative:

$$\begin{array}{ccc} H & \xrightarrow{F} & G \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{h} & \xrightarrow[F_*]{} & \mathfrak{g} \end{array}$$

Proof: See [5, Theorem 3.32]. □

Let M be a differentiable manifold and let G be a Lie group. A smooth map $\alpha : G \times M \longrightarrow M$ satisfying

(1) $\alpha(gh, x) = \alpha(g, \alpha(h, x))$, and

(2) $\alpha(e, x) = x$,

is called a **left group action** of G on M. Sometimes the map α is denoted simply as $\cdot$. The notion of **right group action** is defined similarly. Note that the product of a Lie group G is a left group action. Such an action defines a diffeomorphism $l_g : G \longrightarrow G$ given by $l_g(h) = gh$. Given a smooth form ω on G, we shall say that ω is **left invariant** if and only if $l_g^*(\omega) = \omega$, for every $g \in G$, where l_g^* is the pullback mapping of forms.

Example 1.1.1 (Adjoint Action)**.** The map $\widetilde{\mathrm{Ad}} : G \times G \longrightarrow G$ given by $(g, h) \mapsto ghg^{-1}$ is a left action of G on itself. Note that the identity element e is fixed by the map $\widetilde{\mathrm{Ad}}_g$, for every $g \in G$. Then

$$d\widetilde{\mathrm{Ad}}_g : T_eG \cong \mathfrak{g} \longrightarrow T_eG \cong \mathfrak{g}$$

is a linear isomorphism of $\mathfrak{g}$ onto itself since $d\widetilde{\mathrm{Ad}}_g$ is a diffeomorphism. Hence we have a map

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{Aut}(\mathfrak{g}) \\ g &\longmapsto d\widetilde{\mathrm{Ad}}_g, \end{aligned}$$

called the **adjoint representation** of G.

1.2 Integration on Lie groups

A **volume form** on a smooth n-manifold M is a nowhere vanishing n-form on M. For instance, $dx_1 \wedge \cdots \wedge dx_n$ is a volume form on $\mathbb{R}^n$. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas of a manifold M. This atlas is called **oriented** if all transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ have positive Jacobian determinants everywhere they are defined. A manifold is called **oriented** if it has an oriented atlas, or equivalently, if if has a volume form.

Now we recall how to integrate over a smooth oriented n-manifold M. Let $\{(U_\alpha, \varphi_\alpha)\}$ be an oriented atlas of M. Consider a **partition of unity subordinate to the cover** $\{U_\alpha\}$, i.e., a collection of smooth functions $\rho_\alpha : M \longrightarrow \mathbb{R}$ such that

(1) $\mathrm{supp}(\rho_\alpha) = \overline{\{x \in M : \rho_\alpha(x) \neq 0\}} \subseteq U_\alpha$.

(2) The collection $\{\mathrm{supp}(\rho_\alpha)\}$ is locally finite (every point of M has a neighbourhood which intersects finitely many members of $\{\mathrm{supp}(\rho_\alpha)\}$).

(3) $0 \leq \rho_\alpha \leq 1$ and $\sum_\alpha \rho_\alpha = 1$.

Let ω be a compactly supported n-form in M, where M has the orientation given by the atlas $\{(U_\alpha, \varphi_\alpha)\}$. We define

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega,$$

where $\int_{U_\alpha} = \rho_\alpha\omega$ means $\int_{\varphi_\alpha(U_\alpha)} (\varphi_\alpha^{-1})^*(\rho_\alpha\omega)$ (a Riemann integral). Notice that the previous sum has finitely many summands since $\mathrm{supp}(\omega)$ is compact, and so it is covered by finitely many U_α. It is well known from elementary differential geometry that this definition of integral does not depend on the oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$ and partition of unity either.

The most important theorem about integrals on manifolds is the Stokes's theorem. We state the following special case:

Theorem 1.2.1 (Stokes's Theorem)**.** Let M be an oriented smooth n-manifold. Let ω be a compactly supported $(n-1)$-form on M. Then

$$\int_M d\omega = 0.$$

Proof: See [4, Theorem 14.9]. □

Now we study integration on Lie groups. We shall only consider the case where G is a compact connected Lie group. Recall the following result, which gives rise to a Riemannian structure on every Lie group G.

Theorem 1.2.2. Let G be a compact connected Lie group. Then there exists a symmetric bilinear form $\langle \cdot, \cdot \rangle_K$ on $\mathfrak{g}$, called the Killing form of G, which makes G into a metric space and satisfies

$$\langle \mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y) \rangle_K = \langle X, Y \rangle_K,$$

for every $g \in G$ and $X, Y \in \mathfrak{g}$.

Proposition 1.2.1. Every connected and compact Lie Group is orientable. In fact, G has a unique left-invariant orientation form ω with the property that $\int_G \omega = 1$.

Proof: Let $\{e_1, \ldots, e_n\}$ be an orthogonal basis of $\mathfrak{g}$ with respect to the Killing form $\langle \cdot, \cdot \rangle_K$. Then $\mathfrak{g}^* \cong T_e^*G = \text{span}_{\mathbb{R}}\{e_1^*, \ldots, e_n^*\}$. Let $\omega_e = e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(T_e^*G)$. Note that ω_e is a nonzero alternating n-linear functional on T_eG. Define $\omega : G \longrightarrow \Lambda^n(G)$ by $\omega := l_{g^{-1}}^*(\omega_e)$.

(i) ω is smooth: We only prove the case $n = 2$. The rest follows similarly. Let $X_1, X_2 \in T_gG$. We have

$$\begin{aligned}
(l_{g^{-1}})_* X_1 &= a_1(g)e_1 + a_2(g)e_2, \text{ where } a_1 \text{ and } a_2 \text{ depend smoothly on } g, \\
(l_{g^{-1}})_* X_2 &= b_1(g)e_1 + b_2(g)e_2, \text{ where } b_1 \text{ and } b_2 \text{ depend smoothly on } g, \\
\omega_g(X_1, X_2) &= l_{g^{-1}}^*(e_1^* \wedge e_2^*)(X_1, \ldots, X_2) \\
&= e_1^* \wedge e_2^*((l_{g^{-1}})_* X_1, (l_{g^{-1}})_* X_2) \\
&= e_1^*((l_{g^{-1}})_* X_1) \cdot e_2^*((l_{g^{-1}})_* X_2) - e_1^*((l_{g^{-1}})_* X_2) \cdot e_2^*((l_{g^{-1}})_* X_1), \\
&= a_1(g)b_2(g) - a_2(g)b_1(g) \text{ is a smooth function.}
\end{aligned}$$

Since ω is smooth on a neighbourhood of each point of G, we have that ω is smooth.

(ii) ω is nowhere vanishing: Suppose there exists $g \in G$ such that $\omega_g = 0$. Then $0 = l_{g^{-1}}^*(\omega_e)$. Let $X_1, \ldots, X_n \in T_gG$ be linearly independent vectors. Since $l_{g^{-1}}$ is a diffeomorphism, we have that $(l_{g^{-1}})_* X_1, \ldots, (l_{g^{-1}})_* X_n$ are linearly independent vectors in T_eG. On the other hand, $0 = \omega_e((l_{g^{-1}})_* X_1, \ldots, (l_{g^{-1}})_* X_n)$. Since ω_e is a nonzero alternating n-linear functional on T_eG, we have that the last equality implies that $(l_{g^{-1}})_* X_1, \ldots, (l_{g^{-1}})_* X_n$ are linearly dependent vectors on T_eG, getting a contradiction.

(iii) ω is left-invariant: Let $h \in G$ and $X_1, \ldots, X_n \in T_gG$. We have:

$$\begin{aligned}
(l_h^*\omega)_g(X_1, \ldots, X_n) &= \omega_{l_h(g)}((l_h)_* X_1, \ldots, (l_h)_* X_n) \\
&= (l_{(hg)^{-1}})^* \omega_e((l_h)_* X_1, \ldots, (l_h)_* X_n) \\
&= \omega_e((l_{(hg)^{-1}})_* (l_h)_* X_1, \ldots, (l_{(hg)^{-1}})_* (l_h)_* X_n) \\
&= \omega_e((l_{g^{-1}})_* (l_{h^{-1}})_* (l_h)_* X_1, \ldots, (l_{g^{-1}})_* (l_{h^{-1}})_* (l_h)_* X_n) \\
&= \omega_e((l_{g^{-1}})_* X_1, \ldots, (l_{g^{-1}})_* X_n) \\
&= (l_{g^{-1}})^* \omega_e(X_1, \ldots, X_n) \\
&= \omega_g(X_1, \ldots, X_n).
\end{aligned}$$

$\square$

1.3 The Mapping Degree

Let M and N be two oriented, connected and compact smooth n-manifolds. Let $f : M \longrightarrow N$ be a smooth map and ω an n-form on N with integral 1. Define the **mapping degree** of f to be

$$\deg(f) = \int_M f^*(\omega).$$

Before proving that the degree of f is well defined, i.e., does not depend on the choice of the n-form ω on N with integral 1, we recall the following result:

Theorem 1.3.1 (Poincaré Duality Theorem)**.** If N is a compact and oriented smooth n-manifold, define a map $\mathrm{PD} : \Omega^p(N) \longrightarrow \Omega^{n-p}(N)^*$ by

$$\mathrm{PD}(\omega)(\eta) = \int_N \omega \wedge \eta.$$

Then PD descends to a linear map $\mathrm{PD} : H^p_{\mathrm{dR}}(N) \longrightarrow H^{n-p}_{\mathrm{dR}}(N)^*$, which is a diffeomorphism for each p.

Proof: See [1, Page 65]. □

Proposition 1.3.1. $\deg(f)$ is well defined.

Proof: Let ω and ω' be two n-forms on N such that $\int_N \omega = \int_N \omega' = 1$. Consider the case $p = n$ in the Poincaré Duality Theorem. We have an isomorphism

$$\mathrm{PD} : H^n_{\mathrm{dR}}(N) \longrightarrow H^0_{\mathrm{dR}}(N)^*$$

given by

$$\mathrm{PD}([\alpha])([f]) = \int_N f\alpha,$$

for every $[\alpha] \in H^n_{\mathrm{dR}}(N)$ and every $[f] \in H^0_{\mathrm{dR}}(N)$. Since N is connected, we have that $H^0_{\mathrm{dR}}(N) = \{[f] : f \text{ is constant}\}$. So we get

$$\mathrm{PD}([\omega])([f]) = \int_N f \cdot \omega = f \int_N \omega = f \int_N \omega' = \int_N f \cdot \omega' = \mathrm{PD}([\omega'])([f]),$$

for every $[f] \in H^0_{\mathrm{dR}}(N)$. It follows $\mathrm{PD}([\omega]) = \mathrm{PD}([\omega'])$. Since PD is an isomorphism, we get $[\omega] = [\omega']$, i.e., there exists a smooth $(n-1)$-form η on N such that $\omega = \omega' + d\eta$. Finally, we obtain

$$\begin{aligned}
\int_M f^*(\omega) &= \int_M f^*(\omega' + d\eta) = \int_M (f^*(\omega) + f^*(d\eta)) \\
&= \int_M f^*(\omega') + \int_M f^*(d\eta) = \int_M f^*(\omega') + \int_M df^*(\eta) \\
&= \int_M f^*(\omega') + 0, \text{ by the Stokes's Theorem} \\
&= \int_M f^*(\omega').
\end{aligned}$$

□

Proposition 1.3.2. Let $f : M \longrightarrow N$ be a smooth map between oriented, connected and compact smooth n-manifolds. If f is not surjective, then $\deg(f) = 0$.

Proof: Note that the set $f(M)$ is compact in N. Let $y \in N - f(M)$. Then there exists a neighbourhood V of y such that $V \cap f(M) = \emptyset$. Let B be a closed set contained in V. There exists a bump function for B supported in V, say g. Thus, g is zero outside of V. Let ω be a smooth n-form on N with $\int_N \omega = 1$ and $\eta = g \cdot \omega$. Let $\lambda = \int_N \eta = \int_V g \cdot \omega > 0$. Set $\omega' = \eta/\lambda$. We have $\int_N \omega' = 1$ and $\omega' = 0$ outside of V. Let $(x_1, \ldots, x_n)$ be smooth coordinates for the neighbourhood V. On V we can write $\omega' = \frac{g}{\lambda} dx_1 \wedge \cdots \wedge dx_n$. So we get $f^*(\omega') = \frac{g \circ f}{\lambda} d(x_1 \circ f) \wedge \cdots \wedge d(x_n \circ f) = 0$, since $g(f(x)) = 0$, for every $x \in M$. Hence

$$\deg(f) = \int_M f^*(\eta') = \int_M 0 = 0.$$

□

Given a connected, compact and oriented smooth n-manifold M with orientation ω, for every volume form α and for every $x \in M$ there exists a scalar $\lambda_x \in \mathbb{R}$ such that $\alpha_x = \lambda_x \omega_x$. We say that α is **associated** with the orientation in M if $\lambda_x > 0$, for every $x \in M$. If $f : M \longrightarrow N$ is a smooth map between connected, compact and oriented smooth n-manifolds, and if β is a volume form on M, then $f^*(\beta)$ is a smooth n-form on N. For every $x \in M$, there exists $\lambda_x \in \mathbb{R}$ such that $f^*(\beta)_x = \lambda_x \cdot \alpha_x$. We shall denote $\det(f)(x) = \lambda_x$.

Proposition 1.3.3. Let $f : M \longrightarrow N$ be a smooth map between oriented, connected and compact smooth n-manifolds, and let α and β be two volume forms on M and N, respectively, associated with the orientations on M and N. If y is a point with a finite pre-image $f^{-1}(y) = \{x_1, \ldots, x_n\}$ and x_i is a regular point of f for all i, then

$$\deg(f) = \sum_{i=1}^{n} \operatorname{sign}(\det(f)(x_i)).$$

Proof: Since f is smooth, there exists neighbourhoods U_i and V_i about x_i and y, respectively, such that $f(U_i) \subseteq V_i$. Since M is Hausdorff, we can choose the U_i such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $V = \cap_{i=1}^{n} V_i$. As we did in the proof of the previous proposition, we can construct an n-form β' from β such that $\int_N \beta' = 1$ and $\beta' = 0$ outside of V. Let $f_i := f|_{U_i}$, for each $i = 1, \ldots, n$. Since the sets U_i are disjoint, we get

$$\deg(f) = \int_M f^*(\beta') = \sum_{i=1}^{n} \int_{U_i} (f_i)^*(\beta') = \sum_{i=1}^{n} \deg(f_i).$$

Choose the U_i and V to be diffeomorphic to bounded open subsets of $\mathbb{R}^n$. By the previous equality, it suffices to prove the formula for a smooth map $f : U \longrightarrow V$, where U and V are bounded open subsets of $\mathbb{R}^n$, and a point $y \in \mathbb{R}^n$ with a single pre-image $f^{-1}(y) = \{x\}$. Without loss of generality, we can assume $x = 0$ and $y = 0$ (choose centred coordinate neighbourhoods). In this particular case, the volume forms α and β can be written as

$$\alpha_x = a(x)dx_1 \wedge \cdots \wedge dx_n,$$
$$\beta_x = b(x)dx_1 \wedge \cdots \wedge dx_n.$$

Since α and β are associated with the orientation of $\mathbb{R}^n$, we have $a(x) > 0$ and $b(y) > 0$ for every $x \in U$, and $y \in V$. We have

$$\begin{aligned} f^*(\beta)(x) &= f^*(bdx_1 \wedge \cdots \wedge dx_n)(x) \\ &= (b \circ f)(x)d(x_1 \circ f) \wedge \cdots \wedge d(x_n \circ f) \\ &= (b \circ f)(x) \cdot \det(\mathrm{Jac}(f))dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

Also, $f^*(\beta) = \det(f)\alpha$. So we have

$$\det(f)(x)a(x)dx_1 \wedge \cdots \wedge dx_n = (b \circ f)(x) \cdot \det(\mathrm{Jac}(f))dx_1 \wedge \cdots \wedge dx_n,$$

i.e.,

$$\det(f)(x) = \frac{(b \circ f)(x) \cdot \det(\mathrm{Jac}(f))}{a(x)}.$$

Since $a(x)$ and $b(x)$ are positive, we have $\operatorname{sign}(\det(f)) = \operatorname{sign}(\det(\mathrm{Jac}(f)))$. Now we use the hypothesis that $\det(\mathrm{Jac}(f)) \neq 0$, i.e, f is a local diffeomorphism of a neighbourhood

$U' \subseteq U$ of 0 to a neighbourhood $V' \subseteq V$ of 0. The derived form β' is compactly supported since $\mathrm{supp}(\beta') \subseteq V'$ is closed and bounded in $\mathbb{R}^n$. Since $f : U' \longrightarrow V'$ is a diffeomorphism, we have

$$\mathrm{sign}(\det(\mathrm{Jac}(f))) = \begin{cases} > 0 & \text{if } f \text{ is orientation preserving,} \\ < 0 & \text{if } f \text{ is orientation reversing.} \end{cases}$$

So we get

$$\begin{aligned} \deg(f) &= \int_{\mathbb{R}^n} f^*(\beta') = \mathrm{sign}(\det(\mathrm{Jac}(f))) \int_{\mathbb{R}^n} \beta' \\ &= \mathrm{sign}(\det(\mathrm{Jac}(f))) \\ &= \mathrm{sign}(\det(f)). \end{aligned}$$

□

Chapter 2

Conjugacy Theorems

2.1 Maximal Tori and Weyl Groups

Let G be a compact connected Lie group. A **maximal torus** $T \subseteq G$ is maximal connected abelian subgroup. Every maximal torus is isomorphic to a torus $\mathbb{T}^k = \frac{\mathbb{R}^k}{\mathbb{Z}^k}$. A **generator** of a torus T is an element $t \in T$ such that the set $\{t^n : n \in \mathbb{Z}_{>0}\}$ is dense in T. Generators are characterized by the following result.

Theorem 2.1.1 (Kronecker's Theorem)**.** A vector $v \in \mathbb{R}^k$ represents a generator of $\mathbb{T}^k$ if and only if 1 and the components $v_1, \ldots, v_k$ of v are linearly independent over the rational numbers $\mathbb{Q}$.

Proof: Consider $\mathbb{Z}^k$, $\mathbb{R}^k$ and $\mathbb{T}^k$ as additive groups. We have an exact sequence

$$0 \longrightarrow \mathbb{Z}^k \longrightarrow \mathbb{R}^k \longrightarrow \mathbb{T}^k \longrightarrow 0.$$

Since $\mathbb{R}^k \cong T_0\mathbb{R}^k \cong T_{0\text{mod}\mathbb{Z}^k}\mathbb{T}^k$, the exponential map $\exp : T_{0\text{mod}\mathbb{Z}^k}\mathbb{T}^k \longrightarrow \mathbb{T}^k$ can be identified with the canonical quotient map $\rho : \mathbb{R}^k \longrightarrow \mathbb{T}^k$. By properties of the exponential map (see Proposition 1.1.2), we have the following commutative square:

$$\begin{array}{ccc} \mathbb{R}^k & \xrightarrow{\exp} & \mathbb{T}^k \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{R} & \xrightarrow[\exp]{} & S^1 \end{array}$$

where $f : \mathbb{T}^k \longrightarrow S^1$ is any Lie group homomorphism. This gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{i} & \mathbb{R}^k & \xrightarrow{\exp} & \mathbb{T}^k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_*|_{\mathbb{Z}^k}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[j]{} & \mathbb{R} & \xrightarrow[\exp]{} & S^1 & \longrightarrow & 0 \end{array}$$

where the i and j are inclusions. It follows $f_*(v_1,\ldots,v_k) = \alpha_1 v_1 + \cdots + \alpha_k v_k$, for some $\alpha_1,\ldots,\alpha_k \in \mathbb{Z}$. Now we prove that the following statements are equivalent:

(i) $1, v_1, \ldots, v_k$ are linearly dependent over $\mathbb{Q}$.

(ii) $\sum_{i=1}^k q_i v_i \in \mathbb{Q}$ for some k-tuple $0 \neq (q_1,\ldots,q_k) \in \mathbb{Q}^k$.

(iii) $\sum_{i=1}^k \alpha_i v_i \in \mathbb{Z}$ for some k-tuple $0 \neq (\alpha_1,\ldots,\alpha_k) \in \mathbb{Z}^k$.

(iv) $v\mathrm{mod}\mathbb{Z}^k$ is in the kernel of a nontrivial homomorphism $f : \mathbb{T}^k \longrightarrow S^1$, where $v = (v_1,\ldots,v_k)$.

(v) $v\mathrm{mod}\mathbb{Z}^k$ is not a generator of $\mathbb{T}^k$.

The equivalence $(i) \iff (ii) \iff (iii)$ is clear.

- (iii) $\Longrightarrow$ (iv): Let $v = (v_1,\ldots,v_k)$. Let $f : \mathbb{T}^k \longrightarrow S^1$ be any nontrivial Lie group homomorphism such that $f_*(v_1,\ldots,v_k) = \alpha_1 v_1 + \cdots + \alpha_k v_k$. Then we have

$$\begin{aligned} f(v\mathrm{mod}\mathbb{Z}^k) &= \exp(f_*(v_1,\ldots,v_k)) = \exp(\alpha_1 v_1 + \cdots + \alpha_k v_k) \\ &= e^{i2\pi(\alpha_1 v_1 + \cdots + \alpha_k v_k)} \\ &= 1, \text{ since } \alpha_1 v_1 + \cdots + \alpha_k v_k \in \mathbb{Z}. \end{aligned}$$

  Then, $v\mathrm{mod}\mathbb{Z}^k \in \mathrm{Ker}(f)$.

- (iv) $\Longrightarrow$ (iii): Suppose $v\mathrm{mod}\mathbb{Z}^k \in \mathrm{Ker}(f)$, where $f : \mathbb{T}^k \longrightarrow S^1$ is a nontrivial homomorphism. We have

$$f(\exp(v_1,\ldots,v_k)) = 0 \implies \exp(f_*(v_1,\ldots,v_k)) = 0.$$

  It follows $f_*(v_1,\ldots,v_k) \in \mathbb{Z}$. But $f_*(v_1,\ldots,v_k) = \alpha_1 v_1 + \cdots + \alpha_k v_k$, for some $\alpha_1,\ldots,\alpha_k \in \mathbb{Z}$.

- (iv) $\Longrightarrow$ (v): Let $v\mathrm{mod}\mathbb{Z}^k \in \mathrm{Ker}(f : \mathbb{T}^k \longrightarrow S^1)$, where f is a nontrivial Lie group homomorphism. Then $\mathrm{Ker}(f) \subsetneqq \mathbb{T}^k$, i.e., $\mathrm{Ker}(f)$ is a closed proper Lie subgroup of $\mathbb{T}^k$. We have

$$\overline{\{(v\mathrm{mod}\mathbb{Z}^k)^n : n > 0\}} \subseteq \overline{\mathrm{Ker}(f)} = \mathrm{Ker}(f) \subsetneqq \mathbb{T}^k.$$

  Hence $v\mathrm{mod}\mathbb{Z}^k$ is not a generator.

- (v) $\Longrightarrow$ (iv): If $v\text{mod}\mathbb{Z}^k$ is not a generator then $v\text{mod}\mathbb{Z}^k$ is contained in a proper closed subgroup $H \subsetneq \mathbb{T}^k$. It follows that the quotient $\mathbb{T}^k/H$ is a nontrivial connected abelian Lie group, i.e., a torus. So there is an ismomorphism $\varphi : \mathbb{T}^k/H \longrightarrow \mathbb{T}^m$, for some $m > 0$. Also, we have an isomorphism $\psi : \mathbb{T}^m \longrightarrow S^1 \times \cdots \times S^1$, where $S^1 \times \cdots \times S^1$ has m factors. Let $\rho : \mathbb{T}^k \longrightarrow \mathbb{T}^k/H$ be the quotient projection. Consider the composition $f = \text{pr}_1 \circ \psi \circ \varphi \circ \rho$, where $\text{pr}_1 : S_1 \times \cdots \times S^1 \longrightarrow S^1$ is the projection onto the first factor. Since $v\text{mod}\mathbb{Z}^n \in H$, we have $f(v\text{mod}\mathbb{Z}^k) = 1$. Hence $v\text{mod}\mathbb{Z}^k \in \text{Ker}(f)$, where f is the nontrivial Lie group homomorphism given by the composite map $\text{pr}_1 \circ \psi \circ \varphi \circ \rho$.

□

The **Weyl group** of G associated to a maximal torus T is defined as the quotient group $W(T) = N(T)/T$, where $N(T)$ is the normalizer of T, i.e., $N(T) = \{g \in G/gTg^{-1} = T\}$. We shall prove bellow that any two maximal tori are conjugate. Assume this for a moment to prove the following result.

Proposition 2.1.1. If T and T' are two maximal tori in G, then $W(T)$ and $W(T')$ are isomorphic.

Proof: Since T and T' are conjugate, there exists $g \in G$ such that $T' = gTg^{-1}$. Let $\varphi : N(T') \longrightarrow N(T)/T$ be the map given by $\varphi(x) = g^{-1}xgT$, for every $x \in N(T')$.

(i) φ is well defined: In other words, $g^{-1}xg \in N(T)$. If fact,

$$(g^{-1}xg)T(g^{-1}xg)^{-1} = g^{-1}x(gTg^{-1})x^{-1}g = g^{-1}xT'x^{-1}g = g^{-1}T'g = T.$$

Also, it is clear that φ is a group homomorphism.

(ii) φ is surjective: Let $yT \in N(T)/T$. Then $gyg^{-1} \in N(T')$. In fact,

$$(gyg^{-1})T'(gyg^{-1})^{-1} = gy(g^{-1}T'g)y^{-1}g^{-1} = gyTy^{-1}g^{-1} = gTg^{-1} = T'.$$

Also, $yT = g^{-1}(gyg^{-1})gT = \varphi(gyg^{-1})$.

(iii) $\text{Ker}(\varphi) = T'$: Suppose $\varphi(x) = eT$. Then $(g^{-1}xg)T = eT$, i.e., $g^{-1}xg \in T$. Hence $x = gtg^{-1} \in gTg^{-1} = T'$. Now suppose $x \in T'$. We have $\varphi(x) = g^{-1}xgT = eT$ since $g^{-1}xg \in T$. Therefore, $\text{Ker}(\varphi) = T'$.

By the First Fundamental Theorem of Isomorphisms, we have $N(T)/\text{Ker}(\varphi) \cong \text{Im}(\varphi)$, i.e. $N(T)/T \cong N(T')/T'$. □

Note that the normalizer $N(T)$ acts on T via conjugation

$$\begin{array}{c} N(T) \times T \longrightarrow T \\ (n,t) \mapsto ntn^{-1} \end{array}$$

For each $n \in N(T)$, we have the adjoint automorphism $\widetilde{\mathrm{Ad}}_n : T \longrightarrow T$ given by $\widetilde{\mathrm{Ad}}_n(t) = ntn^{-1}$. So we get a continuos map $N(T) \longrightarrow \mathrm{Aut}(T)$. Since T is isomorphic to some $\mathbb{T}^k$, we have $\mathrm{Aut}(T) \cong \mathrm{Aut}(\mathbb{T}^k)$. Recall from the proof of the Kronecker's theorem that given an automorphism $\varphi : \mathbb{T}^k \longrightarrow \mathbb{T}^k$ one can construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{i} & \mathbb{R}^k & \xrightarrow{\exp} & \mathbb{T}^k & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_*|_{\mathbb{Z}^k}} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow[i]{} & \mathbb{R}^k & \xrightarrow[\exp]{} & \mathbb{T}^k & \longrightarrow & 0 \end{array}$$

Lemma 2.1.1. Let $\Phi : \mathrm{Aut}(\mathbb{T}^k) \longrightarrow \mathrm{Aut}(\mathbb{Z}^k)$ be the map $\varphi \mapsto \varphi_*|_{\mathbb{Z}^k}$. Then Φ is an isomorphism.

Proof: It is clear that Φ is a group homomorphism. Let $\{e_1, \ldots, e_k\}$ denote the canonical basis of $\mathbb{R}^k$.

(i) Φ is injective: Suppose $\varphi_*|_{\mathbb{Z}^k} = \mathrm{Id}$. Then, $\varphi_* = \mathrm{Id}$. In fact,

$$\begin{aligned} \varphi_*(v_1, \ldots, v_k) &= \varphi_*(v_1 e_1 + \cdots + v_k e_k) \\ &= v_1 \varphi_*(e_1) + \cdots + v_k \varphi_*(e_k) \\ &= v_1 e_1 + \cdots + v_k e_k, \text{ since each } e_j \in \mathbb{Z}^k \\ &= (v_1, \ldots, v_k). \end{aligned}$$

So we get $\varphi(v \mathrm{mod} \mathbb{Z}^k) = \varphi \circ \exp(v_1, \ldots, v_k) = \exp(v_1, \ldots, v_k) = v \mathrm{mod} \mathbb{Z}^k$. Hence $\varphi = \mathrm{Id}$.

(ii) Φ is surjective: Let $F : \mathbb{Z}^k \longrightarrow \mathbb{Z}^k$ be an automorphism. Define $\widetilde{F} : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ by

$$\widetilde{F}(v_1 e_1 + \cdots + v_k e_k) = v_1 F(e_1) + \cdots + v_k F(e_k).$$

Now define $\varphi : \mathbb{T}^k \longrightarrow \mathbb{T}^k$ by

$$\varphi((v_1, \ldots, v_k) \mathrm{mod} \mathbb{Z}^k) = \exp \circ \widetilde{F}(v_1, \ldots, v_k).$$

We show φ is well defined. Suppose $(v_1,\dots,v_k)\mathrm{mod}\mathbb{Z}^k = (w_1,\dots,w_k)\mathrm{mod}\mathbb{Z}^k$. Then $v_i = w_i + m_i$, for some $m_i \in \mathbb{Z}$, for every $i = 1,\dots,k$. We have

$$\begin{aligned}
\exp\circ\widetilde{F}(v_1,\dots,v_k) &= \exp\circ\widetilde{F}(v_1e_1+\cdots+v_ke_k)\\
&= \exp(v_1F(e_1)+\cdots+v_kF(e_k))\\
&= \exp(w_1F(e_1)+\cdots+w_kF(e_k)+m_1F(e_1)+\cdots+m_kF(e_k))\\
&= \exp(w_1F(e_1)+\cdots+w_kF(e_k))\\
&= \exp(\widetilde{F}(w_1,\dots,w_k)).
\end{aligned}$$

The map φ is surjective since exp and $\widetilde{F}$ are. Suppose $\varphi(v\mathrm{mod}\mathbb{Z}^k) = 0\mathrm{mod}\mathbb{Z}^k$. Then $\exp(\widetilde{F}(v)) = 0$. Since exp is the canonical quotient map $\mathbb{R}^k \longrightarrow \mathbb{R}^k/\mathbb{Z}^k$, we have that $\widetilde{F}(v) \in \mathbb{Z}^k$. So $v \in \mathbb{Z}$, i.e., $v\mathrm{mod}\mathbb{Z}^k = 0\mathrm{mod}\mathbb{Z}^k$. Hence φ is surjective. Therefore, φ is an automorphism. Also, $\Phi(\varphi) = \varphi_*|_{\mathbb{Z}^k} = \widetilde{F}|_{\mathbb{Z}^k} = F$.

□

Theorem 2.1.2. The Weyl group associated to T is finite.

Proof: Let N_0 be the connected component of the identity in $N(T)$. We shall prove $N_0 = T$. Then $W(T) = N(T)/N_0$. This quotient is compact since $N(T)$ is compact. We shall also show that $N(T)/N_0$ is discrete. Since every compact and discrete space is finite, we have $W(T) = N(T)/N_0$ is finite.

(i) $N_0 = T$: By the previous lemma, we can consider the adjoint action of $N(T)$ on T as a continuous map $f : N(T) \longrightarrow \mathrm{Aut}(T) \cong \mathrm{Gl}(k,\mathbb{Z})$. Since N_0 is connected we have that $f(N_0)$ is connected in $\mathrm{Gl}(k,\mathbb{Z})$. On the other hand, $\mathrm{Gl}(k,\mathbb{Z})$ is a discrete space, it follows $f(N_0) = \{\mathrm{Id}\}$. We get $ntn^{-1} = t$, for every $n \in N_0$ and every $t \in T$. In other words, N_0 acts trivially on T. Let $\alpha : \mathbb{R} \longrightarrow N_0$ be a one-parameter group, i.e., a continuous group homomorphism. Since $\alpha(\mathbb{R})$ and T are connected, we have that $\alpha(\mathbb{R})\cdot T$ is connected in N_0. Then $\alpha(\mathbb{R})\cdot T$ is a connected abelian subgroup of G (i.e., a torus) containing T. By maximality of T, we have $\alpha(\mathbb{R})\cdot T = T$. Hence $\alpha(\mathbb{R}) \subseteq T$. Now, the image $\alpha(\mathbb{R})$ covers a neighbourhood of the identity in N_0. Notice that N_0 is closed since it is a connected component. By the Closed Submanifold Theorem, we have that N_0 is a Lie group. It follows that $\alpha(\mathbb{R})$ generates N_0, i.e., $N_0 = \bigcup_{n\in\mathbb{N}} \alpha(\mathbb{R})^n$, (see [5, Proposition 3.18]). Then, $T \subseteq N_0 \subseteq T$. Therefore $N_0 = T$.

(ii) $N(T)/N_0$ is discrete: We have to show that every point in $N(T)/N_0$ is open. Let xN_0 be a point of $N(T)/N_0$. We know xN_0 is open in $N(T)/N_0$ is and only if $\rho^{-1}(xN_0)$ is open in N, where $\rho : N(T) \longrightarrow N(T)/N_0$ is the canonical quotient map. We have

$$\rho^{-1}(xN_0) = \{g \in N(T) : gN_0 = xN_0\} = \{g \in N(T) : g = xn,\ \exists! n \in N_0\}.$$

Note that $\rho^{-1}(xN_0)$ is diffeomorphic to N_0. On the other hand, N_0 is open in $N(T)$ since it is a connected component of a locally path connected space. Hence $\rho^{-1}(xN_0)$ is open.

□

2.2 Conjugacy Theorems

We need some more structure before proving the conjugacy theorems. Let H be a closed Lie subgroup of G. As a consequence of the Quotient Manifold Theorem (see [4, Theorem 9.16]), the quotient group G/H is a smooth manifold of dimension $\dim(G) - \dim(H)$, and the canonical quotient map $\rho : G \longrightarrow G/H$ is smooth. As we did in Theorem 1.2.1, we shall construct a nowhere vanishing left invariant form on G/T.

Theorem 2.2.1. Let H and G as above. If H is connected, then G/H is orientable.

Proof: Let τ_e be a nonzero alternating $(n-k)$-linear functional on the tangent space $T_{eH}G/H$. Define a map $\tau : G/H \longrightarrow \Lambda^{n-k}(G/H)$ by $\tau(gH) = L^*_{g^{-1}}(\tau_e)$, where $L_{g^{-1}} : G/H \longrightarrow G/H$ is the map $L_{g^{-1}}(xH) = g^{-1}xH$. We only proof that τ is well defined. The rest of the proof (τ is smooth, nowhere vanishing and left invariant) is similar to the proof of Theorem 1.2.1. Suppose $gH = g'H$. Then $g' = gh$ for some $h \in H$. We have to show $L^*_{(gh)^{-1}}(\tau_e) = L^*_{g^{-1}}(\tau_e)$. Note that $L^*_{(gh)^{-1}}(\tau_e) = L^*_{h^{-1}g^{-1}}(\tau_e) = L^*_{g^{-1}}L^*_{h^{-1}}(\tau_e)$. Then $L^*_{(gh)^{-1}}(\tau_e) = L^*_{g^{-1}}(\tau_e)$ if and only if $\tau_e = L^*_{h^{-1}}(\tau_e)$. Also, note that $L^*_{h^{-1}}(\tau_e) \in \Lambda^{n-k}(T^*_{eH}G/H)$, i.e., $L^*_{h^{-1}}$ is an alternating $(n-h)$-linear functional on $T_{eH}G/H$. If we follow the construction done in the proof of Theorem 1.2.1, we have that τ_e is the determinant function. From linear algebra we know $L^*_{h^{-1}}(\tau_e) = \lambda(h)\tau_e$, for some $\lambda(h) \in \mathbb{R}$ (see [3, Theorem 2, page 152]). By properties of pullbacks, we have that λ is a group homomorphism from H to the multiplicative group $\mathbb{R}^* = \mathbb{R} - \{0\}$. Moreover, λ is a continuous map since $L^*_{h^{-1}}$ is smooth. Since H is connected and $\lambda(e) = 1$, we have that $\lambda(h) > 0$ for every $h \in H$. Now suppose that there exists $h \in H$ such that $\lambda(h) \neq 1$. We can assume $\lambda(h) > 1$. Then $\lambda(h)^n = \lambda(h^n) \in \lambda(H) = [a,b]$. Note that $\lambda(H)$ is a closed interval since H is connected and compact (it is a closed subgroup of a compact group G). Since $\lambda(h) > 1$, the fact that $\lambda(h)^n \in [a,b]$ for every $n \in \mathbb{N}$ is a contradiction. It follows $\lambda(h) = 1$ for every $h \in H$. Hence $L^*_{h^{-1}}(\tau_e) = \tau_e$. □

We shall denote the forms ω and τ on G and G/H by dg and $d(gH)$, respectively. Let $\mathfrak{g}$ be the Lie algebra if G. Let T be a maximal torus of G. Since T is a Lie subgroup of G, we have that $\mathfrak{t}$, the Lie algebra of T, is a Lie subalgebra of $\mathfrak{g}$. Consider the Killing form $\langle \cdot, \cdot \rangle_K$ on $\mathfrak{g}$. We know this product is invariant under the action of the adjoint representation $\mathrm{Ad}(g)$ for all $g \in G$. Let $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{t}$ with respect to $\langle \cdot, \cdot \rangle_K$. Notice that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. We have the following result:

Proposition 2.2.1. The adjoint map restricted to the torus $\mathrm{Ad}|_T$ acts trivially on every vector in $\mathfrak{t}$, and non-trivially on every nonzero vector in $\mathfrak{m}$. Moreover, for every $t \in T$, the map $\mathrm{Ad}(t) : \mathfrak{g} \longrightarrow \mathfrak{g}$ preserves $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$.

Proof: Let $V \in \mathfrak{t}$. We show that $\mathrm{Ad}(t)(V) = V$, for every $t \in T$. We know $\mathrm{Ad}(t)(V) = d\widetilde{\mathrm{Ad}}(t)(V)$. Let f be a smooth function about e. Then $d\widetilde{\mathrm{Ad}}(t)(V)(f) = V(f \circ \mathrm{Ad}(t))$. Let α be an integral curve of V passing through e. We have

$$\begin{aligned} d\widetilde{\mathrm{Ad}}(t)(V)(f) &= \alpha'(0)(f \circ \widetilde{\mathrm{Ad}}(t)) = \left.\frac{d}{ds}\right|_{s=0} (f \circ \widetilde{\mathrm{Ad}}(t) \circ \alpha(s)) \\ &= \left.\frac{d}{ds}\right|_{s=0} (f(t\alpha(s)t^{-1})) \\ &= \left.\frac{d}{ds}\right|_{s=0} f(\alpha), \text{ since } t, \alpha(s) \in T \text{ and } T \text{ is abelian} \\ &= V(f). \end{aligned}$$

Hence $\mathrm{Ad}(t)(V) = V$. Now suppose that V is a vector in $\mathfrak{m}$ such that $\mathrm{Ad}(t)(V) = V$. Let α be the maximal integral curve of V passing through e. Consider $\beta(s) = t\alpha(s)t^{-1}$. We have

$$\begin{aligned} V(f) &= \mathrm{Ad}(t)(V)(f) = d\widetilde{\mathrm{Ad}}(t)(V)(f) = \left.\frac{d}{ds}\right|_{s=0} (f \circ \widetilde{\mathrm{Ad}}(t) \circ \alpha(s)) \\ &= \left.\frac{d}{ds}\right|_{s=0} (f \circ \beta(s)) = \beta'(0)(f). \end{aligned}$$

Then β is a maximal integral curve of V. By the uniqueness of α, we get $t\alpha(s) = \alpha(s)t$, i.e., $\alpha(\mathbb{R})$ commutes with T. So $\alpha(R) \cdot T$ is a connected abelian group containing T.

By maximality of T, we get $T = \alpha(\mathbb{R}) \cdot T$. Thus, $\alpha(R) \subseteq T$. It follows $V \in \mathfrak{t}$. Hence $V \in \mathfrak{t} \cap \mathfrak{m} = \{0\}$ and so $V = 0$.

Finally, we prove that $\mathrm{Ad}(t)$ preserves $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$. We already know that $\mathrm{Ad}(t)(\mathfrak{t}) \subseteq \mathfrak{t}$. It is only left to show that $\mathrm{Ad}(t)(\mathfrak{m}) \subseteq \mathfrak{m}$. Let $X \in \mathfrak{m}$ and $Y \in \mathfrak{t}$. We have

$$\langle Y, \mathrm{Ad}(t)(X) \rangle_K = \left\langle \mathrm{Ad}(t^{-1})(Y), \mathrm{Ad}(t^{-1})(\mathrm{Ad}(t)(X)) \right\rangle_K = \left\langle \mathrm{Ad}(t^{-1})(Y), X \right\rangle_K = 0.$$

Hence $\mathrm{Ad}(t)(X) \in \mathfrak{m}$. □

We have that $\mathfrak{m}$ is an invariant subspace of $\mathrm{Ad}(t)$, for every $t \in T$. Then we can define an action of T on $\mathfrak{m}$, denoted

$$\begin{aligned} \mathrm{Ad}_{G/T} : T &\longrightarrow \mathrm{Aut}(\mathfrak{m}) \\ t &\mapsto \mathrm{Ad}(t)|_{\mathfrak{m}}. \end{aligned}$$

We know that G/T is a smooth manifold of dimension $n - k$, where $n = \dim(G)$ and $k = \dim(T)$. Since the canonical quotient map $\rho : G \longrightarrow G/T$ is smooth, it induces a map $\rho_* : \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \longrightarrow T_{eT}G/T$. It is known that ρ_* maps $\mathfrak{m}$ isomorphically onto $T_{eT}G/T$. We shall identify $\mathfrak{m}$ with $T_{eT}G/T$ via this isomorphism. Since T is a Lie group, we can find a volume form dt which is left invariant. Then $d(gT)dt$ is a nowhere vanishing left invariant form on $G/T \times T$. Note that $(d(gT)dt)(eT, e)$ and $dg(e)$ are nonzero alternating n-linear functionals on $\mathfrak{g}$. We know that, up to multiplication by a constant, there is a unique alternating n-linear functional on $\mathfrak{t} \oplus \mathfrak{m}$. So we get $(d(gT)dt)(eT, e) = c \cdot dg(e)$, where $c > 0$ or $c < 0$. Replacing $-d(gT)$ by $d(gT)$ if necessary, we may assume that $c > 0$. From now on, we consider the manifolds $G/T \times T$ and G with the orientations given by $d(gT)dt$ and dg. Consider the map $q : G/T \times T \longrightarrow G$ given by $(gT, t) \mapsto gtg^{-1}$. Note that q is a well defined smooth map. In order to prove the conjugacy theorems, we need the following result:

Lemma 2.2.1 (Main Lemma)**.** Let G be a connected and compact Lie group and T a maximal torus in G. Then the map q has mapping degree $\deg(q) = |W(T)|$, where $|W(T)|$ is the order of the Weyl group associated to T.

Theorem 2.2.2 (First Conjugacy Theorem)**.** In a connected compact Lie group G, every element is conjugate to an element in any fixed maximal torus.

Proof: By the previous lemma, $\deg(q) = |W(T)| \neq 0$, and Proposition 1.3.2 implies q is surjective. Let $g \in G$. Then there exists $hT \in G/T$ and $t \in T$ such that $g = q(hT, t) = hth^{-1}$, i.e., g is conjugate to t, where $t \in T$. □

Theorem 2.2.3 (Second Conjugacy Theorem)**.** Any two maximal tori in a connected compact Lie group G are conjugate.

Proof: Let T and T' be two maximal tori in G. We have to show $T' = gTg^{-1}$ for some $g \in G$. Let t be a generator of T, which exists by the Kronecker's theorem. By the previous theorem, there exists $g \in G$ such that $gtg^{-1} \in T'$. It follows $gt^n g^{-1} = (gtg^{-1})^n \in T'$, for every $n > 0$. Consider the set $\{gt^n g^{-1} : n > 0\}$. Note that $g\overline{\{t^n : n > 0\}}g^{-1}$ is a closed set containing $\{gt^n g^{-1} : n > 0\}$. Then $\overline{\{gt^n g^{-1} : n > 0\}} \subseteq g\overline{\{t^n : n > 0\}}g^{-1}$. The other inclusion follows similarly. Hence

$$\overline{\{gt^n g^{-1} : n > 0\}} = g\overline{\{t^n : n > 0\}}g^{-1} = gTg^{-1}.$$

Since $\overline{\{gt^n g^{-1} : n > 0\}} \subseteq T'$, we get $gTg^{-1} \subseteq T'$, i.e., $T \subseteq g^{-1}T'g$, where $g^{-1}T'g$ is a torus in G. By maximality of T, we get $T = g^{-1}T'g$, i.e., $gTg^{-1} = T'$. □

The proof of the main lemma is quite difficult and technical. We prove the following two helper lemmas. Using them, the proof of the main lemma follows easily.

Lemma 2.2.2. $\operatorname{sign}(\det(q)(gT,t)) = \operatorname{sign}(\det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T}))$, where $\mathrm{Id}_{G/T}$ is the identity on $\mathfrak{m}$.

Proof: Let $V_1, \ldots, V_n$ be vector fields on $G/T \times T$. Then

$$\begin{aligned}
&q^* dg(V_1(gT,t), \ldots, V_n(gT,t)) = \\
&= dg(q_*(V_1(gT,t)), \ldots, q_*(V_n(gT,t))) \\
&= l^*_{q(gT,t)^{-1}} dg(q_*(V_1(gT,t)), \ldots, q_*(V_n(gT,t))), \text{ since } dg \text{ is left invariant} \\
&= dg((l_{q(gT,t)^{-1}})_* q_*(V_1(gT,t)), \ldots, (l_{q(gT,t)^{-1}})_* q_*(V_n(gT,t))) \\
&= dg((l_{q(gT,t)^{-1}} \circ q)_*(V_1(gT,t)), \ldots, (l_{q(gT,t)^{-1}} \circ q)_*(V_n(gT,t))).
\end{aligned}$$

Since $d(gT)dt$ is left invariant we also have

$$\begin{aligned}
&(d(gT)dt)(V_1(gT,t), \ldots, V_n(gT,t)) = \\
&= (d(gT)dt)((l_{(gT,t)^{-1}})_*(V_1(gT,t)), \ldots, (l_{(gT,t)^{-1}})_*(V_n(gT,t))),
\end{aligned}$$

Notice that

$$\begin{aligned}
l_{q(gT,t)^{-1}} \circ q(gT,t) &= L_{(gtg^{-1})^{-1}}(gtg^{-1}) = (gtg^{-1})^{-1} \cdot (gtg^{-1}) = e, \\
l_{(gT,t)^{-1}}(gT,t) &= e,
\end{aligned}$$

then the arguments of the expressions above are elements of $T_eG \cong \mathfrak{t} \oplus \mathfrak{m}$.

Now we use the equality $(d(gT)dt)(eT,e) = cdg(e)$, where $c > 0$. Write $W_i(gT,t) = (L_{(gT,t)^{-1}})_* V_i(gT,t)$, for simplicity. We obtain

$$(d(gT)dt)(V_1(gT,t),\dots,V_n(gT,t)) = c\cdot dg(W_1(gT,t),\dots,W_n(gT,t)).$$

Thus,

$$\begin{aligned}
&q^*dg(V_1(gT,t),\dots,V_n(gT,t)) = \\
&= dg((l_{q(gT,t)^{-1}}\circ q)_* V_1(gT,t),\dots,(l_{q(gT,t)^{-1}}\circ q)_* V_n(gT,t)) \\
&= dg((l_{q(gT,t)^{-1}}\circ q)_*(l_{(gT,t)^{-1}})_*^{-1} W_1(gT,t),\dots,(l_{q(gT,t)^{-1}}\circ q)_*(l_{(gT,t)^{-1}})_*^{-1} W_n(gT,t)) \\
&= dg((l_{q(gT,t)^{-1}}\circ q)_*(l_{(gT,t)})_* W_1(gT,t),\dots,(l_{q(gT,t)^{-1}}\circ q)_*(l_{(gT,t)})_* W_n(gT,t)) \\
&= dg((l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)})_* W_1(gT,t),\dots,(l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)})_* W_n(gT,t)) \\
&= \det(\mathrm{Jac}(l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)})_*) dg(W_1(gT,t),\dots,W_n(gT,t)).
\end{aligned}$$

It follows

$$q^*dg = \frac{\det(\mathrm{Jac}(l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)})_*)}{c}\cdot d(gT)dt.$$

By definition of $\det(q)(gT,t)$, we obtain

$$\det(q)(gT,t) = \frac{\det(\mathrm{Jac}(l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)})_*)}{c}.$$

We compute $l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)}$:

$$\begin{aligned}
l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)}(hT,s) &= l_{q(gT,t)^{-1}}\circ q(ghT,ts) = l_{q(gT,t)^{-1}}((gh)(ts)(gh)^{-1}) \\
&= l_{gt^{-1}g^{-1}}(ghtsh^{-1}g^{-1}) = gt^{-1}g^{-1}ghtsh^{-1}g^{-1} \\
&= gt^{-1}htsh^{-1}g^{-1}.
\end{aligned}$$

On the other hand, $\widetilde{\mathrm{Ad}}(g)(\widetilde{\mathrm{Ad}}(t^{-1})(h)sh^{-1}) = \widetilde{\mathrm{Ad}}(g)(t^{-1}htsh^{-1}) = gt^{-1}htsh^{-1}g^{-1}$. Hence $l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)}(hT,s) = \widetilde{\mathrm{Ad}}(g)(\widetilde{\mathrm{Ad}}(t^{-1})(h)sh^{-1})$. Now consider the function $f : G/T\times T \longrightarrow G$ given by $f(hT,s) = \widetilde{\mathrm{Ad}}(t^{-1})(h)sh^{-1}$. So we get

$$\begin{aligned}
l_{q(gT,t)^{-1}}\circ q\circ l_{q(gT,t)} &= \widetilde{\mathrm{Ad}}(g)\circ f, \\
(l_{q(gT,t)^{-1}}\circ q\circ l_{q(gT,t)})_* &= (\widetilde{\mathrm{Ad}}(g)\circ f)_* = \widetilde{\mathrm{Ad}}(g)_*\circ f_* \\
&= d(\widetilde{\mathrm{Ad}}(g))\circ f_* = \mathrm{Ad}(g)\circ f_*, \\
\det(q)(gT,t) &= \frac{1}{c}\det(\mathrm{Jac}(l_{q(gT,t)^{-1}}\circ q\circ l_{(gT,t)})_*) \\
&= \frac{1}{c}\det(\mathrm{Ad}(g)\circ f_*) \\
&= \frac{1}{c}\det(\mathrm{Ad}(g))\cdot\det(f_*).
\end{aligned}$$

Recall that $\mathrm{Ad}(g)$ is orthogonal with respect to the Killing form $\langle\cdot,\cdot\rangle_K$. Then $\det(\mathrm{Ad}(g)) = \pm 1$. Since G is connected and $\det(\mathrm{Ad}(e)) = 1$, we get $\det(\mathrm{Ad}(g)) = 1$, for every $g \in G$. Hence $\det(q)(gT,t) = \frac{1}{c}\det(f_*)$. In order to compute the matrix of f_*, it suffices to determine the matrices of $f_*|_{\mathfrak{t}}$ and $f_*|_{\mathfrak{m}}$, since $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ and

$$f_* = \begin{pmatrix} f_*|_{\mathfrak{t}} & 0 \\ 0 & f_*|_{\mathfrak{m}} \end{pmatrix}.$$

Let $V \in \mathfrak{t}$ and let $\alpha : \mathbb{R} \longrightarrow G$ be the maximal integral curve of V passing through e. Then $\beta(r) = (eT, \alpha(r))$ is also an integral curve of V. Let g be a smooth function about e. We have

$$\begin{aligned}
f_*|_{\mathfrak{t}}(V)(g) &= f_* \circ \beta_* \left(\frac{d}{dr}\bigg|_{r=0} \right)(g) = (f \circ \beta)_* \left(\frac{d}{dr}\bigg|_{r=0} \right)(g) \\
&= \frac{d}{dr}\bigg|_{r=0} (g \circ f \circ \beta(r)) = \frac{d}{dr}\bigg|_{r=0} (g(f(eT, \alpha(r)))) \\
&= \frac{d}{dr}\bigg|_{r=0} (g(\widetilde{\mathrm{Ad}}(t^{-1})(e)\alpha(r)e^{-1})) = \frac{d}{dr}\bigg|_{r=0} (g(\alpha(r))) \\
&= \alpha_* \left(\frac{d}{dr}\bigg|_{r=0} \right)(g) \\
&= V(g).
\end{aligned}$$

We get $f_*|_{\mathfrak{t}}(V) = V$ for every $V \in \mathfrak{t}$, i.e., $f_*|_{\mathfrak{t}} = \mathrm{Id}_T$. Let $V \in \mathfrak{m}$ and let $\alpha : \mathbb{R} \longrightarrow G$ be the maximal integral curve of V passing through e. Then $\beta(r) = (\alpha(r)T, e)$ is an integral curve of V. We have

$$\begin{aligned}
f_*|_{\mathfrak{m}}(V)(g) &= V(g \circ f) = \beta'(0)(g \circ f) \\
&= \frac{d}{dr}\bigg|_{r=0} (g \circ f \circ \beta(r)) = \frac{d}{dr}\bigg|_{r=0} g \circ f(\alpha(r)T, e) \\
&= \frac{d}{dr}\bigg|_{r=0} g(\widetilde{\mathrm{Ad}}(t^{-1})(\alpha(r))e\alpha(r)^{-1}) \\
&= \frac{d}{dr}\bigg|_{r=0} g(\widetilde{\mathrm{Ad}}(t^{-1})(\alpha(r))\alpha(-r)) \\
&= \gamma_* \left(\frac{d}{dr}\bigg|_{r=0} \right)(g),
\end{aligned}$$

where $\gamma(r) = \widetilde{\mathrm{Ad}}(t^{-1})(\alpha(r)) \cdot \alpha(-r)$. Since G is a Lie group, the differential of the product is the sum of the differentials. Then $\gamma_* = d\widetilde{\mathrm{Ad}}(-t) \circ \alpha_* - \alpha_*$.

We have

$$\begin{aligned}
f_*|_{\mathfrak{m}}(V)(g) &= \gamma_*\left(\frac{d}{dr}\bigg|_{r=0}\right)(g) \\
&= d\widetilde{\mathrm{Ad}}(t^{-1})\circ\alpha_*\left(\frac{d}{dr}\bigg|_{r=0}\right)(g) - \alpha_*\left(\frac{d}{dr}\bigg|_{r=0}\right)(g) \\
&= \mathrm{Ad}(t^{-1})V(g) - V(g) = \left[\mathrm{Ad}_{G/T}(t^{-1})(V) - \mathrm{Id}_{G/T}(V)\right](g).
\end{aligned}$$

Hence $f_*|_{\mathfrak{m}} = \mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T}$. Finally, we get

$$f_* = \begin{pmatrix} \mathrm{Id}_T & 0 \\ 0 & \mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T} \end{pmatrix},$$

and then

$$\begin{aligned}
\det(q)(gT,t) &= \frac{1}{c}\det(f_*) \\
&= \frac{1}{c}\det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T})\cdot\det(\mathrm{Id}_T) \\
&= \frac{1}{c}\det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T}).
\end{aligned}$$

Since $c > 0$, we obtain $\mathrm{sign}(\det(q)(gT,t)) = \mathrm{sign}(\det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T}))$. □

Lemma 2.2.3. There exists a generator t of T such that:

(i) $|q^{-1}(t)| = |W(T)|$;

(ii) $\mathrm{Ad}_{G/T}(t^{-1})$ has no real eigenvalues. Consequently, $\dim(G/T)$ is even.

(iii) $\det(q) > 0$ at each point of $q^{-1}(t)$.

Proof:

(i) Let t be any generator of T. Let $(gT,s)\in q^{-1}(t)$. Then $gsg^{-1} = t$, i.e., $g^{-1}tg = s \in T$. As we did in the proof of the Second Conjugacy Theorem, we have that $\overline{\{g^{-1}t^ng : n>0\}} = g^{-1}Tg$. Since $g^{-1}tg \in T$, we have $\overline{\{g^{-1}t^ng : n>0\}} \subseteq T$. It follows $T \subseteq gTg^{-1}$. Also, gTg^{-1} is a torus since it is a connected abelian Lie subgroup of G. So we get $T = gTg^{-1}$, by maximality of T. Hence $g \in N(T)$. Define a map $\varphi : q^{-1}(t) \longrightarrow W(T)$ by $\varphi(gT,s) = gT$. It is clear that φ is well defined. We also show it is a bijection.

- φ is injective: Suppose $\varphi(g_1T, s_1) = \varphi(g_2T, s_2)$. Then $g_1T = g_2T$ and hence $g_2 = g_1t'$, for some $t' \in T$. Since T is abelian, we have

$$s_1 = g_1^{-1}(g_2s_2g_2^{-1})g_1 = g_1^{-1}g_1t's_2(g_1t')^{-1}g_1 = t's_2(t')^{-1}g_1^{-1}g_1 = t's_2(t')^{-1} = s_2.$$

  Hence $(g_1T, s_1) = (g_2T, s_2)$.

- φ is surjective: Let $gT \in W(T)$ and $s = g^{-1}tg$. Then $s \in T$ since $g \in N(T)$. Also, $(gT, s) \in q^{-1}(t)$ since $q(gT, s) = gsg^{-1} = t$. Moreover, $gT = \varphi(gT, s)$. Hence φ is onto.

Since φ is a bijection, we get $|q^{-1}(t)| = |W(T)|$.

(ii) Recall that $\mathrm{Ad}_{G/T}(t^{-1})$ is an orthonormal linear transformation on $\mathfrak{m}$ with respect to $\langle\cdot,\cdot\rangle_K$. Then the real eigenvalues of $\mathrm{Ad}_{G/T}(t^{-1})$ (if any) must be ±1. By the Kronecker's Theorem $t_0 = t^2$ is also a generator of T. Replace t by t_0 and suppose that $\mathrm{Ad}_{G/T}(t^{-1})$ has a real eigenvalue $\lambda = \pm1$. We have $\mathrm{Ad}_{G/T}(t^{-1})(V) = \lambda V$, where V is an eigenvector associated to λ. Also,

$$\begin{aligned}\mathrm{Ad}_{G/T}(t_0^{-1})(V) &= \mathrm{Ad}_{G/T}(t^{-1}t^{-1})(V) = \mathrm{Ad}_{G/T}(t^{-1}) \cdot \mathrm{Ad}_{G/T}(t^{-1})(V)\\ &= \mathrm{Ad}_{G/T}(t^{-1})(\lambda V) = \lambda^2 V = V.\end{aligned}$$

Hence 1 is a real eigenvalue of $\mathrm{Ad}_{G/T}(t_0^{-1})$. It follows $\mathrm{Ad}_{G/T}(t_0^{-n})(V) = V$, for every $n > 0$. Since t_0 is a generator of T, we have $T = \overline{\{t_0^{-n} : n > 0\}}$. Consider the set $E_V = \{s \in T : \mathrm{Ad}_{G/T}(s)(V) = V\}$. Note that E_V is a closed set containing $\{t_0^{-n} : n > 0\}$. It follows $T = \overline{\{t_0^{-n} : n > 0\}} \subseteq E_V$ and so $\mathrm{Ad}_{G/T}(s)(V) = V$, for every $s \in T$. On the other hand, Proposition 2.2.1 states that $\mathrm{Ad}_{G/T}(s)$ acts nontrivially on every nonzero vector of $\mathfrak{m}$, getting a contradiction since $\mathrm{Ad}_{G/T}(s)(V) = V$ and $V \neq 0$. Therefore, the automorphism $\mathrm{Ad}_{G/T}(t^{-1}) : \mathfrak{m} \cong T_{eT}G/T \longrightarrow \mathfrak{m} \cong T_{eT}G/T$ has no real eigenvalues. It follows the characteristic polynomial of $\mathrm{Ad}_{G/T}(t^{-1})$ has even degree, since every polynomial of odd degree has at least one real root. Hence $\dim(G/T) = \dim(T_{eT}G/T)$ is even.

(iii) Let $(gT, s) \in q^{-1}(t)$. By part (ii), $\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T}$ has no real eigenvalues. Let $A = \mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T}$. Suppose that A has negative determinant. Let

$$p(x) = \det(xI - A) = x^n + \cdots + a_1x + a_0$$

be the characteristic polynomial of A. Note that $a_0 = (-1)^n\det(A) = \det(A)$, since n is even. Thus, $p(0) < 0$. On the other hand, $\lim_{x\to+\infty} p(x) = +\infty$. So there exists $a > 0$ such that $p(a) > 0$. By the Intermediate Value Theorem, there exists $b \in (0, a)$ such that $p(b) = 0$, i.e., b is a real eigenvalue of A, getting a contradiction. Therefore, $\det(\mathrm{Ad}_{G/T}(t^{-1}) - \mathrm{Id}_{G/T}) > 0$. By the previous lemma, we have $\det(q)(gT, s) > 0$.

□

Proof of the Main Lemma. We know that $q^*dg = \det(q)(d(gT)dt)$. By the previous lemma, let t be a generator of T satisfying (i), (ii) and (iii). Then $q^{-1}(t) = \{(g_1T, s_1), \ldots, (g_mT, s_m)\}$, where $m = |W(T)|$. By Proposition 1.3.3, we have $\deg(q) = \sum_{i=1}^{m} \operatorname{sign}(\det(q)(g_iT, s_i))$. By (iii), we have $\operatorname{sign}(\det(q)(g_iT, s_i)) = 1$. Hence

$$\deg(q) = \sum_{i=1}^{m} 1 = |W(T)|.$$

□

Now we give some examples of maximal tori and Weyl groups. If you are interested in the details, see [2, Chapter 4, Section 3].

Example 2.2.1.

(1) Consider the group U(1) of unitary $n \times n$ matrices. Let $\Delta(n) \subseteq \mathrm{U}(1)$ be the subgroup of diagonal matrices

$$D = \begin{pmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_n \end{pmatrix}, \quad z_i \in S^1,$$

and let $\mathrm{S}\Delta(n) = \Delta(n) \cap \mathrm{SU}(n)$ be the subgroup of diagonal matrices as above with $z_1 \cdots z_n = 1$. The groups $\Delta(n)$ and $\mathrm{S}(n)$ are maximal tori in $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, respectively. The Weyl group of the unitary group $\mathrm{U}(n)$ is the symmetric group S_n.

(2) Let $\mathrm{SO}(2n)$ be the orthogonal special group, i.e., the orthogonal matrices whose determinant in 1. The subgroup $T(n) = \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)$ (n times) is a maximal torus in $\mathrm{SO}(2n)$ and $\mathrm{SO}(2n+1)$. Let G_n be the group of permutations σ of the set $\{-n, \ldots, -1, 1, \ldots, n\}$ for which $\sigma(-i) = -\sigma(i)$. The Weyl group of $\mathrm{SO}(2n+1)$ is G_n. Also, the Weyl group of $\mathrm{SO}(2n)$ is SG_n, the subgroup of G_n consisting of even permutations.

(3) Now consider the symplectic group $\mathrm{Sp}(n)$ of unitary matrices of the form $\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}$.

We have a canonical inclusion $\mathrm{U}(n) \longrightarrow \mathrm{Sp}(n)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}$. Let $T(n)$ be the image of the torus $\Delta(n)$ under this inclusion. Then $T(n)$ is a maximal torus in $\mathrm{Sp}(n)$. The Weyl group of $\mathrm{Sp}(n)$ is G_n.

2.3 Some consequences of the Conjugacy Theorems

We collect a few consequences of the Second Conjugacy Theorem. From now on, any discussion of the maximal torus or operation of the Weyl group shall refer to a particular torus chosen once and for all. As we saw above, any other choice leads to conjugate tori and isomorphic Weyl groups. Hence, the following result do not depend of the choice of a particular torus.

Corollary 2.3.1. The exponential map of a compact connected Lie group G is surjective.

Proof: Let $g \in G$, there exists $h \in G$ such that $g \in hTh^{-1}$. Notice that hTh^{-1} is a torus in G. If there is no torus T' such that $hTh^{-1} \subseteq T'$, then hTh^{-1} is a maximal torus. Now if there exists a maximal torus T' such that $hTh^{-1} \subseteq T'$, then $g \in T'$. In any case, g is in a maximal torus in G. Let T be a maximal torus containing g. The exponential map $\exp : \mathfrak{t} \longrightarrow T$ is surjective since it can be identified with the canonical quotient map $\rho : \mathbb{R}^k \longrightarrow \mathbb{R}/\mathbb{Z}^k \cong T$, where $k = \dim(T)$. Since ρ is surjective, so is $\exp : \mathfrak{t} \longrightarrow T$. Hence there exists $V \in \mathfrak{t}$ such that $g = \exp(V)$. □

The **centralizer** of a subgroup $H \subseteq G$ is the subgroup $Z(H) = \{g \in G : gh = hg \ \forall h \in H\}$. The **center** is the centralizer of the entire group G.

Corollary 2.3.2. Let G be a compact connected Lie group and T its maximal torus.

(i) If $S \subseteq G$ is a connected abelian subgroup, then $Z(S)$ is the union of the maximal tori containing S.

(ii) $Z(T) = T$.

(iii) The center of G is the intersection of all maximal tori in G. In particular, $Z(G)$ is contained in any maximal torus.

Proof:

(i) Note that $\overline{S}$ is a connected abelian subgroup of G, i.e., a torus. Let $g \in Z(S)$. Then $gs = sg$, for every $s \in S$. It folows $gs = sg$, for every $s \in \overline{S}$. Hence $Z(S) \subseteq Z(\overline{S})$. The other inclusion is obvious. We get $Z(S) = Z(\overline{S})$. So we may assume that $S = \overline{S}$ is a torus. Let $x \in Z(S)$ and let $\langle x, S\rangle$ be the subgroup generated by x and S. Let $B = \overline{\langle x, S\rangle}$. Then B is a compact abelian subgroup of G. Let B_0 denote the connected component of the identity of B. We have that B_0 is a torus, since it is connected and abelian. Consider the quotient B/B_0. We show that xB_0 generates B/B_0. Consider $yB_0 \in B/B_0$. If $y \in \langle x, S\rangle$ then $y = x^n s_1^{k_1} \cdots s_m^{k_m}$, for some $n, m \in \mathbb{Z}$. Since S is connected and the identity e is in S, we have $S \subseteq B_0$ since B_0 is a component. Then, $yB_0 = x^n B_0$. It follows xB_0 generates B/B_0. As we did in the proof of Theorem 2.1.2, we have that B/B_0 is finite. Then B/B_0 is a finite cyclic group and hence there exists $l \in \mathbb{N}$ such that $B/B_0 \cong \mathbb{Z}_l$. It follows $B \cong B_0 \times \mathbb{Z}_l$. It is known that a compact Lie group contains a dense cyclic subgroup if and only if is isomorphic to $\mathbb{T}^k \times \mathbb{Z}_l$, for some $k, l \in \mathbb{N}$. (See [2, Corollary 4.14]). Thus, B contains a dense cyclic subgroup $\{g^n : n \in \mathbb{Z}\}$. We get $B = \overline{\{g^n : n \in \mathbb{Z}\}}$. Let T be a maximal torus containing g. Then $g^n \in T$ for every $n \in \mathbb{Z}$ and so $B \subseteq T$. Then $x \in T$ and $S \subseteq T$. We obtain $Z(S) \subseteq \bigcup\{T : T \text{ is a maximal torus and } T \supseteq S\}$. The other inclusion follows easily since T is abelian.

(ii) $Z(T) = \bigcup\{T' : T' \text{ is a maximal torus and } T' \supseteq T\} = T$, since T is maximal.

(iii) Suppose x is in the center of G. There exists a maximal torus T in G such that $x \in T$. Let T' be any other maximal torus in G. By the Second Conjugacy Theorem, there exists $g \in G$ such that $T' = gTg^{-1}$. Then $x = gg^{-1}x = gxg^{-1} \in T'$. Hence x is in any maximal torus of G, and $Z(G) \subseteq \bigcap\{T : T \text{ is a maximal torus in } G\}$. Now suppose that x is in every maximal torus. Let $g \in G$. There exists a maximal torus T such that $g \in T$. Since $x, g \in T$ and T is abelian, we get $xg = gx$. Then $x \in Z(G)$ and the other inclusion follows.

□

The Weyl group $W(T)$ of G acts on T via the map product $xT \cdot t = xtx^{-1}$. Consider the group homomorphism $\rho : W(T) \longrightarrow \mathrm{Aut}(T)$ given by

$$\rho(xT) : T \longrightarrow T \qquad \rho(xT)(t) = xtx^{-1}.$$

Corollary 2.3.3. The Weyl group acts **effectively** on the maximal torus, i.e., the homomorphism ρ is injective.

Proof: Suppose $\rho(xT) = \mathrm{Id}_T$. Then $xtx^{-1} = t$, i.e., $xt = tx$ for every $t \in T$. Hence $x \in Z(T) = T$ and so $xT = eT$. □

Recall that the **orbit** of $t \in T$ under the action ρ is the set $W(T)_t = \{gT \cdot t : gT \in W(T)\}$.

Corollary 2.3.4. Two elements of the maximal torus are conjugate in G if and only if they lie in the same orbit under the action of the Weyl group.

Proof: Let $x, y \in T$ such that $y = gxg^{-1}$, for some $g \in G$. Let $Z(x)$ and $Z(y)$ be the centralizers of x and y. Consider the map $\widetilde{\mathrm{Ad}}_g : G \longrightarrow G$. Let $h \in Z(x)$. We have

$$\begin{aligned}\widetilde{\mathrm{Ad}}_g(h)y &= (ghg^{-1})y = (gh)(g^{-1}y) = (gh)(xg^{-1}) = g(hx)g^{-1} \\ &= g(xh)g^{-1} = (gx)(hg^{-1}) = (yg)(hg^{-1}) = y(ghg^{-1}) \\ &= y\widetilde{\mathrm{Ad}}_g(h).\end{aligned}$$

Then $\widetilde{\mathrm{Ad}}_g(h) \in Z(y)$. We have $\widetilde{\mathrm{Ad}}_g(T) \subseteq Z(y)$ since $T \subseteq Z(x)$. Let $T' = \widetilde{\mathrm{Ad}}_g(T)$. We have that T and T' are maximal tori in the connected component $Z(y)_0$ of the identity of $Z(y)$. Then there exists $z \in Z(y)_0$ such that $zT'z^{-1} = T$, i.e.,

$$T = \widetilde{\mathrm{Ad}}_z(T') = \widetilde{\mathrm{Ad}}_z \circ \widetilde{\mathrm{Ad}}_g(T) = \widetilde{\mathrm{Ad}}_{zg}(T).$$

We obtain $T = zgT(zg)^{-1}$ and hence $zg \in N(T)$. Thus, $zgT \in W(T)$. Moreover,

$$(zgT) \cdot x = (zg)x(zg)^{-1} = z(gxg^{-1})z^{-1} = zyz^{-1} = y, \text{ since } z \in Z(y).$$

Therefore, x and y are in the same orbit. Now suppose that $y = (gT) \cdot x$, for some $gT \in W(T)$. Then $y = gxg^{-1}$ and hence x and y are conjugate. $\square$

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