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THE FREYD'S ADJOINT FUNCTOR THEOREM

January 2011

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Introduction

Adjoint functors are pairs of functors, $G : \mathcal{D} \longrightarrow \mathcal{C}$ and $F : \mathcal{C} \longrightarrow \mathcal{D}$, which stand in a particular relationship with one another, called an adjunction. Specifically, (F, G) is an adjunction if there exists a natural isomorphism $\theta : \mathrm{Hom}_{\mathcal{D}}(F(\cdot), \cdot) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\cdot, G(\cdot))$. Under these conditions, G is said to have a left adjoint, namely F. Of course, not every functor admits a left adjoint. For instance, if X is a set having two or more elements, then the functor $X \times - : \mathbf{Set} \longrightarrow \mathbf{Set}$ does not have a left adjoint.

A natural question that comes to us is under which conditions it is possible to determine the existence of a left adjoint F for a given functor G. Peter Freyd answered that question, for the particular case where the category $\mathcal{D}$ is complete. If $\mathcal{D}$ is a complete category, then the functors with left adjoints can be characterized by the adjoint functor theorem:

Theorem. Given a small and complete category $\mathcal{D}$, a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint if and only if it preserves all limits and satisfies the following
Solution Set Condition. For each object $C \in \mathrm{Ob}(\mathcal{C})$ there is a set I and an I-indexed family of arrows $f_i : C \longrightarrow G(D_i)$ such that every arrow $h : C \longrightarrow G(D)$ can be written as a composite $h = G(t) \circ f_i$ for some index i and some $t : D_i \longrightarrow D$.

In these notes we shall give a prove of this result due to Saunders Mac Lane. First, we study universal arrows and characterize them as initial objects in certain comma categories. Then we study the concept of a limit for a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$. Limits give rise to a special type of categories known as complete categories. In these categories there exists a lot of universal constructions, such as products, equalizers and pullbacks. We begin the last chapter recalling the notion of adjunction. Adjoint functors have a deep relation with limits, in fact, every functor having a left adjoint preserves limits. Before proving the Freyd's theorem, we first study the case of the existence of an initial object in a category and then use the fact that each universal arrow defined by the unit of a left adjoint is an initial object in a suitable comma category.

Chapter 1

Universal Arrows

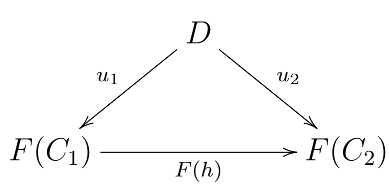
1.1 Comma categories and universal arrows

A comma category is a construction in category theory, introduced in 1963 by F. W. Lawvere, which provides another way of looking at morphisms: instead of simply relating objects of a category to one another, morphisms become objects in their own right. We shall see there are certain guarantees about the existence of limits and colimits in the context of comma categories.

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and $D \in \mathrm{Ob}(\mathcal{D})$. We define $(D \downarrow F)$ the **comma category of objects F-under D** as follows:

(1) the objects of $(D \downarrow F)$ are all pairs (u, C) where $C \in \mathrm{Ob}(\mathcal{C})$ and $u : D \longrightarrow F(C)$ is an arrow of $\mathcal{D}$;

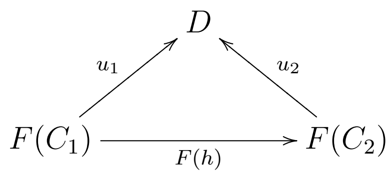
(2) the arrows of $(D \downarrow F)$, $h : (u_1, C_1) \longrightarrow (u_2, C_2)$, are arrows $h : C_1 \longrightarrow C_2$ of $\mathcal{C}$ making the following triangle commute:



Similarly, the **comma category $(F \downarrow D)$ of objects F-over D** is defined as follows:

(1) the objects of $(F \downarrow D)$ are all pairs (u, C) where $C \in \mathrm{Ob}(\mathcal{C})$ and $u : F(C) \longrightarrow D$ is an arrow of $\mathcal{D}$;

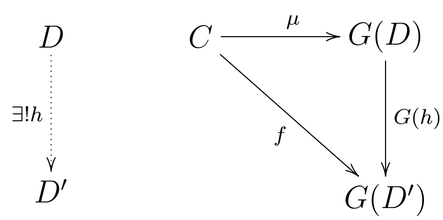
(2) the arrows of $(F \downarrow D)$, $h : (u_1, C_1) \longrightarrow (u_2, C_2)$, are arrows $h : C_1 \longrightarrow C_2$ of $\mathcal{C}$ such that the triangle



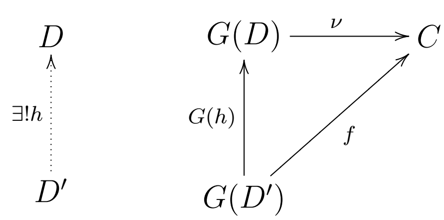
commutes.

Example 1.1.1. Let $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ be the forgetful functor and X a set. An object of $(X \downarrow U)$ is a function $X \longrightarrow U(G)$ from X into the underlying set of G, where G is a group.

If $G : \mathcal{D} \longrightarrow \mathcal{C}$ is a functor and $C \in \mathrm{Ob}(\mathcal{C})$, a **universal arrow from** C **to** G is a pair (D, μ) consisting of an object D of $\mathcal{D}$ and an arrow $\mu : C \longrightarrow G(D)$ of $\mathcal{C}$, such that for every arrow $f : C \longrightarrow G(D')$ of $\mathcal{C}$ there is a unique arrow $h : D \longrightarrow D'$ of $\mathcal{D}$ such that the diagram



commutes. The arrow μ is also called G**-free**. Similarly, a **universal arrow from** G **to** C is a pair (D, ν) where D is an object of $\mathcal{D}$ and $\nu : G(D) \longrightarrow C$ is an arrow of $\mathcal{C}$, such that for every arrow $f : G(D') \longrightarrow C$ of $\mathcal{C}$ there exists a unique arrow $h : D' \longrightarrow D$ of $\mathcal{D}$ such that the diagram



commutes. The arrow ν is also called G**-cofree**.

Proposition 1.1.1.

(1) If (D_1, μ_1) and (D_2, μ_2) are universal arrows from C to G, then D_1 and D_2 are isomorphic.

(2) If (D_1, ν_1) and (D_2, ν_2) are universal arrows from G to C, then D_1 and D_2 are isomorphic.

Proof: We only prove (1). Part (2) can be proven in a similar way.

Since (D_1, μ_1) is a universal arrow from C to G, there exists a unique arrow $h : D_1 \longrightarrow D_2$ of $\mathcal{D}$ such that $G(h) \circ \mu_1 = \mu_2$. Similarly, there exists a unique arrow $h' : D_2 \longrightarrow D_1$ such that $G(h') \circ \mu_2 = \mu_1$. So we get

$$\begin{aligned} G(h') \circ (G(h) \circ \mu_1) &= \mu_1 \\ (G(h') \circ (G(h)) \circ \mu_1 &= \mu_1 \\ (G(h' \circ h)) \circ \mu_1 &= \mu_1. \end{aligned}$$

On the other hand, id_{D_1} is the only arrow of $\mathcal{D}$ satisfying $G(\mathrm{id}_{D_1}) \circ \mu_1 = \mu_1$. Hence $h' \circ h = \mathrm{id}_{D_1}$. Similarly, $h \circ h' = \mathrm{id}_{D_2}$. Therefore, $h : D_1 \longrightarrow D_2$ is an isomorphism. □

From the definitions of comma categories and universal arrows, the following proposition is immediate.

Proposition 1.1.2.

(1) $\mu : C \longrightarrow G(D)$ is a universal arrow from C to G if and only if (D, μ) is an initial object in the comma category $(C \downarrow G)$.

(2) $\nu : G(D) \longrightarrow C$ is a universal arrow from G to C if and only if (D, ν) is a terminal object in the comma category $(G \downarrow C)$.

1.2 Limits and colimits

The abstract notion of a limit captures the essential properties of universal constructions such as products, equalizers and pullbacks. The dual notion of a colimit generalizes constructions such as disjoint unions, direct sums, coproducts and pushouts.

Let $\mathcal{J}$ be a small category, i.e., $\mathrm{Ob}(\mathcal{J})$ and $\mathrm{Hom}(\mathcal{J})$ are sets. Let $\mathcal{C}$ be a category. Consider the category $[\mathcal{J}, \mathcal{C}]$ whose objects are all functors $F : \mathcal{J} \longrightarrow \mathcal{C}$ and whose arrows are all natural transformations $\alpha : F \longrightarrow G$. Define a functor $\Delta : \mathcal{C} \longrightarrow [\mathcal{J}, \mathcal{C}]$ as follows:

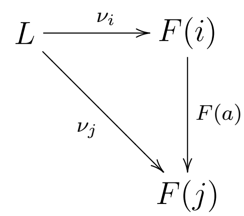
(1) $\Delta(C) : \mathcal{J} \longrightarrow \mathcal{C}$ is the constant functor

$$\begin{aligned} \Delta(i) &= C && \text{for every } i \in \mathrm{Ob}(\mathcal{J}), \text{ and} \\ \Delta(a) &= \mathrm{id}_C && \text{for every arrow } a : i \longrightarrow j \text{ of } \mathcal{J}; \end{aligned}$$

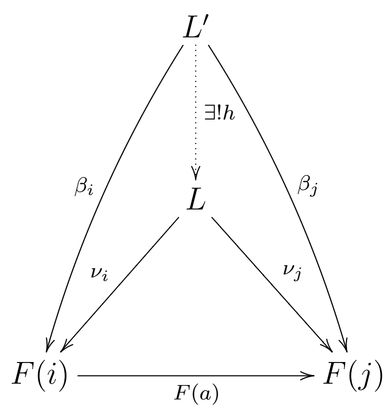
(2) for every arrow $f : C_1 \longrightarrow C_2$ of $\mathcal{C}$, $\Delta(f) : \Delta(C_1) \longrightarrow \Delta(C_2)$ is the natural transformation defined as the $\mathcal{J}$-indexed set

$$\Delta(f) = \{\Delta(f)_i = f : C_1 \longrightarrow C_2\}_{i \in \mathrm{Ob}(\mathcal{J})}.$$

The functor Δ is known as the **diagonal functor**. A **limit** for a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ is a universal arrow (L, ν) from Δ to F. In this situation, $\nu : \Delta(L) \longrightarrow F$ is a natural transformation, i.e., a $\mathcal{J}$-indexed family of arrows $\nu_i : L \longrightarrow F(i)$ of $\mathcal{C}$, with $i \in \mathrm{Ob}(\mathcal{J})$, such that the triangle



commutes, for every arrow $a : i \longrightarrow j$ of $\mathcal{J}$. Moreover, if $\beta : \Delta(L') \longrightarrow F$ is another natural transformation, i.e., a $\mathcal{J}$-indexed family of arrows $(\beta_i : L' \longrightarrow F(i))_{i \in \mathrm{Ob}(\mathcal{J})}$ of $\mathcal{C}$ satisfying $F(a) \circ \beta_i = \beta_j$ for every arrow $a : i \longrightarrow j$, then there exists a unique arrow $h : L' \longrightarrow L$ of $\mathcal{C}$ such that $\beta = \nu \circ \Delta(h)$, i.e., $\beta_i = \nu_i \circ h$ for every $i \in \mathrm{Ob}(\mathcal{J})$. In pictures, we have the commutative diagram



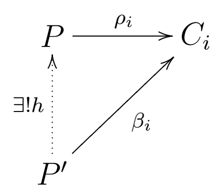
By Proposition 1.1.1, the object L is unique up to isomorphisms.

Example 1.2.1.

(1) **Terminal objects:** Let $\mathcal{J}$ be the empty category. Then any $F : \mathcal{J} \longrightarrow \mathcal{C}$ is the empty functor. Thus, every pair $(L, \nu : \Delta(L) \longrightarrow F)$ is just the object L. If the previous pair is the limit for F, then for any other pair $(L', \beta : \Delta(L') \longrightarrow F)$, i.e., any other object L', there exists a unique arrow $h : L' \longrightarrow L$ of $\mathcal{C}$. Hence, L is a **terminal object** in $\mathcal{C}$.

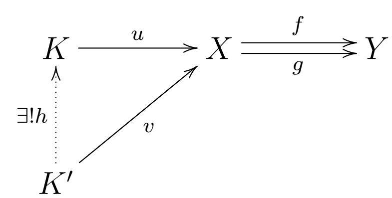
(2) **Products:** Let $\mathcal{J} = I$ be a discrete small category, i.e., a small category whose only arrows are the identity arrows. A functor $F : I \longrightarrow \mathcal{C}$ is just a I-indexed set of objects $\{C_i = F(i)\}_{i \in I}$. Suppose $(P, \rho : \Delta(P) \longrightarrow F)$ is the limit for F, then ρ is just a family of arrows $\rho_i : P \longrightarrow C_i$ of $\mathcal{C}$. If $(\beta_i : P' \longrightarrow C_i)_{i \in I}$ is another family of arrows of $\mathcal{C}$, i.e., a

natural transformation $\beta : \Delta(P') \longrightarrow F$, then there exists a unique arrow $h : P' \longrightarrow P$ of $\mathcal{C}$ such that $\rho_i \circ h = \beta_i$, for every $i \in I$. In pictures, we have the commutative triangle



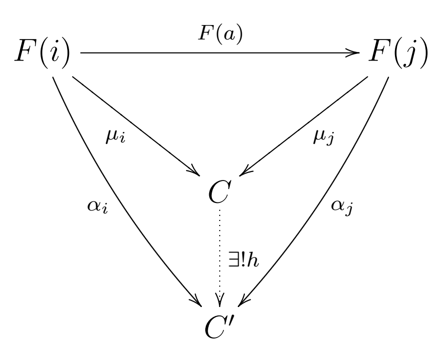
In this case, $(P, (\rho_i : P \longrightarrow C_i)_{i \in I})$ is called the **product** of the family $\{C_i\}_{i \in I}$ and is denoted $P = \prod_i C_i$. The arrows ρ_i are called projections.

(3) **Equalizers:** Let $\mathcal{J} = \{1, 2\}$ be a two-object category having two parallel arrows $a, b : 1 \longrightarrow 2$. Let $X = F(1)$, $Y = F(2)$, $f = F(a)$ and $g = F(b)$. Suppose there exists the limit $(K, \nu : \Delta(K) \longrightarrow F)$ for the functor F, which is just the pair of arrows $\{f, g\}$. Then ν is just a pair of arrows $u = \nu_1 : K \longrightarrow X$ and $\nu_2 : K \longrightarrow Y$ satisfying $f \circ u = g \circ u = \nu_2$. Moreover, if $v : K' \longrightarrow X$ is another arrow satisfying $f \circ v = g \circ v$, then there exists a unique arrow $h : K' \longrightarrow K$ such that $v = u \circ h$. In pictures, we have the commutative diagram



In this case, the arrow $u : K \longrightarrow X$ is called the **equalizer** of f and g.

Dually, given a small category $\mathcal{J}$, a **colimit** for a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ is a universal arrow (C, μ) from F to Δ, i.e., a family of arrows $\mu_i : F(i) \longrightarrow C$ of $\mathcal{C}$ such that the $\mu_i = \mu_j \circ F(a)$, for every arrow $a : i \longrightarrow j$ of $\mathcal{J}$; and if $(\alpha_i : F(i) \longrightarrow C')_{i \in \mathrm{Ob}(\mathcal{J})}$ is another family of arrows of $\mathcal{C}$ satisfying $\alpha_j \circ F(a) = \alpha_i$ for every arrow $a : i \longrightarrow j$, then there exists a unique arrow $h : C \longrightarrow C'$ of $\mathcal{C}$ such that $\alpha_i = h \circ \mu_i$. In pictures, we have the commutative diagram



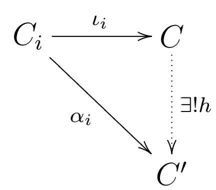
By Proposition 1.1.1, the object C is unique up to isomorphisms.

Example 1.2.2.

(1) **Initial objects:** If $F : \mathcal{J} \longrightarrow \mathcal{C}$ is the empty functor, then the colimit for F is an initial object in $\mathcal{C}$.

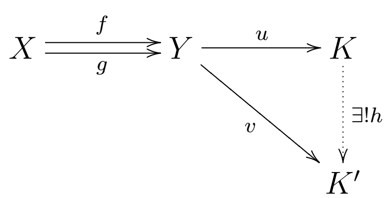
(2) **Coproducts:** The coproduct of a family $\{C_i\}_{i \in I}$ of objects of $\mathcal{C}$, where I is a set, is the colimit for the functor $F : I \longrightarrow \mathcal{C}$, where the set I can be considered as a discrete category, satisfying $F(i) = C_i$ for each $i \in I$. Specifically, it is an object C of $\mathcal{C}$ and an I-indexed family of arrows $\iota_i : C_i \longrightarrow C$ such that if $\alpha_i : C_i \longrightarrow C'$ is another I-indexed family of arrows then there exists a unique arrow $h : C \longrightarrow C'$ such that $\alpha_i = h \circ \iota_i$, for every $i \in I$.

In pictures, we have the commutative diagram



The object C is denoted $C = \coprod_i C_i$ and the arrows ι_i are called inclusions.

(3) **Coequalizers:** The coequalizer of two parallel arrows $f, g : X \longrightarrow Y$ of $\mathcal{C}$ is the colimit of the functor $F : \mathcal{J} \longrightarrow \mathcal{C}$, where $\mathcal{J} = \{1, 2\}$ is a two-object category with two parallel arrows $a, b : 1 \longrightarrow 2$, satisfying $X = F(1)$, $Y = F(2)$, $f = F(a)$ and $g = F(b)$. It is an arrow $u : Y \longrightarrow K$ such that $u \circ f = u \circ g$, and if $v : Y \longrightarrow K'$ is another arrow of $\mathcal{C}$ satisfying $v \circ f = v \circ g$ then there exists a unique arrow $h : K \longrightarrow K'$ of $\mathcal{C}$ such that $v = h \circ u$. In pictures, we have the commutative diagram



A category $\mathcal{C}$ is said to be **complete** if every functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ has a limit in $\mathcal{C}$, where $\mathcal{J}$ is a small category. Dually, $\mathcal{C}$ is **cocomplete** if every functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ has a colimit in $\mathcal{C}$.

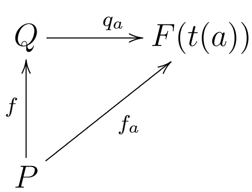
Theorem 1.2.1. A category $\mathcal{C}$ is complete if and only if it has products and equalizers. Dually, $\mathcal{C}$ is cocomplete if and only if it has coproducts and coequalizers.

Proof: By Example 1.2.1, if $\mathcal{C}$ is complete then it has products and equalizers. Now suppose that $\mathcal{C}$ is a category in which there exist products and equalizers. Let $\mathcal{J}$ be a small category and $F : \mathcal{J} \longrightarrow \mathcal{C}$ a functor. For every arrow a of $\mathcal{J}$, we denote $o(a)$ the domain of a and $t(a)$ the codomain of a. Since $\mathrm{Ob}(\mathcal{J})$ and $\mathrm{Hom}(\mathcal{J})$ are sets, we can take products

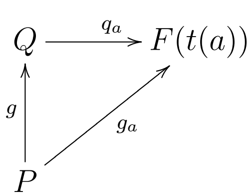
$$\left[P = \prod_{i \in \mathrm{Ob}(\mathcal{J})} F(i), (p_i : P \longrightarrow F(i))_{i \in \mathrm{Ob}(\mathcal{J})}\right],$$

$$\left[Q = \prod_{a \in \mathrm{Hom}(\mathcal{J})} F(t(a)), (q_i : Q \longrightarrow F(t(a)))_{a \in \mathrm{Hom}(\mathcal{J})}\right].$$

Let $f_a = p_{t(a)} : P \longrightarrow F(t(a))$ for each $a \in \mathrm{Hom}(\mathcal{J})$. Since the arrows q_a are universals, we have that there exists a unique arrow $f : P \longrightarrow Q$ such that the triangle



commutes. Similarly, there exists a unique arrow $g : P \longrightarrow Q$ and a commutative triangle

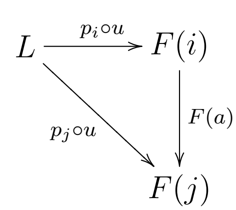


where g_a is the composite arrow $P \xrightarrow{p_{o(a)}} F(o(a)) \xrightarrow{F(a)} F(t(a))$. Let $u : L \longrightarrow P$ be the equalizer of f and g. We shall prove that $(L, (p_i \circ u : L \longrightarrow F(i))_{i \in \mathrm{Ob}(\mathcal{J})})$ is the limit of $F : \mathcal{J} \longrightarrow \mathcal{C}$.

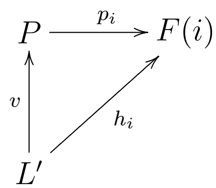
(1) Let $a : i \longrightarrow j$ be a morphism of $\mathcal{J}$. We have

$$\begin{aligned} F(a) \circ (p_i \circ u) &= (F(a) \circ p_i) \circ u = g_a \circ u = (q_a \circ g) \circ u = q_a \circ (g \circ u) \\ &= q_a \circ (f \circ u) = (q_a \circ f) \circ u = f_a \circ u = p_j \circ u, \end{aligned}$$

i.e., the following triangle commutes



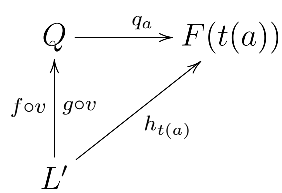
(2) Let $(h_i : L' \longrightarrow F(i))_{i \in \mathrm{Ob}(\mathcal{J})}$ be another family of arrows of $\mathcal{C}$ satisfying $F(a) \circ h_i = h_j$, for each arrow $a : i \longrightarrow j$ of $\mathcal{J}$. Since the arrows p_i are universal, there exists a unique arrow $v : L' \longrightarrow P$ of $\mathcal{C}$ such that the triangle



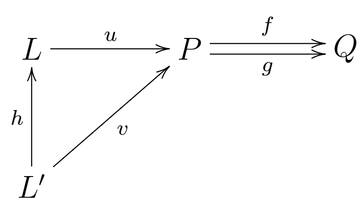
commutes. On the other hand,

$$\begin{aligned} q_a \circ (f \circ v) &= (q_a \circ f) \circ v = f_a \circ v = p_{t(a)} \circ v = h_{t(a)} \\ q_a \circ (g \circ v) &= (q_a \circ g) \circ v = g_a \circ v = (F(a) \circ p_{o(a)}) \circ v \\ &= F(a) \circ (p_{o(a)} \circ v) = F(a) \circ h_{o(a)} = h_{t(a)}. \end{aligned}$$

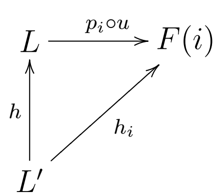
We have that the triangle



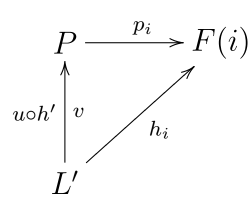
commutes. But $h_{t(a)}$ factors uniquely through q_a, so we get $f \circ v = g \circ v$. Since the arrow u is universal, there exists a unique arrow $h : L' \longrightarrow L$ such that the diagram



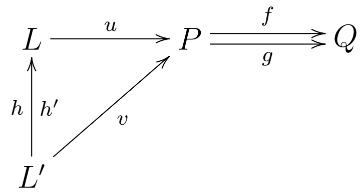
commutes. Moreover $(p_i \circ u) \circ h = p_i \circ (u \circ h) = p_i \circ v = h_i$, i.e. the triangle



commutes. It is only left to show that h is the only arrow of $\mathcal{C}$ satisfying $h_i = (p_i \circ u) \circ h$, for every $i \in \mathrm{Ob}(\mathcal{J})$. Suppose there exists another arrow $h' : L' \longrightarrow L$ such that $(p_i \circ u) \circ h' = h_i$. Then we have the following commutative triangle



It follows $v = u \circ h'$, since h_i factors uniquely through p_i. Similarly, from the commutative diagram



we can conclude that $h = h'$.

$\square$

Example 1.2.3. By the previous theorem, one can get some complete and cocomplete categories, for example **Set**, **Grp** and $\mathbf{Vct}_{\mathbb{K}}$.

Chapter 2

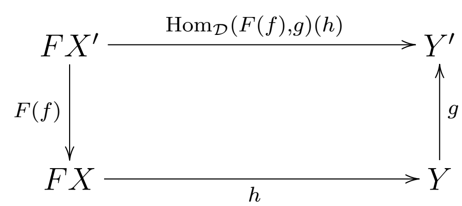
Limits and Adjunction

2.1 Adjoint functors

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ be functors. Consider the product category $\mathcal{C}^{\text{op}} \times \mathcal{D}$. If we assume that $\mathcal{C}$ and $\mathcal{D}$ are small, then $\text{Hom}_{\mathcal{D}}(FX, Y)$ and $\text{Hom}_{\mathcal{C}}(X, GY)$ are objects of **Set**, for every pair $(X, Y) \in \text{Ob}(\mathcal{C}^{\text{op}} \times \mathcal{D})$. If $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, X')$ and $g \in \text{Hom}_{\mathcal{D}}(Y, Y')$, then we define the map $\text{Hom}_{\mathcal{D}}(F(f), g) : \text{Hom}_{\mathcal{D}}(FX, Y) \longrightarrow \text{Hom}_{\mathcal{D}}(FX', Y)$ as the function

$$\text{Hom}_{\mathcal{D}}(F(f), g)(h) = g \circ h \circ F(f),$$

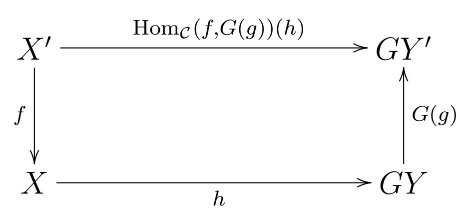
for every arrow $h : FX \longrightarrow Y$ of $\mathcal{D}$.



Similarly, we can define a function $\text{Hom}_{\mathcal{C}}(f, G(g)) : \text{Hom}_{\mathcal{C}}(X, GY) \longrightarrow \text{Hom}_{\mathcal{C}}(X', GY')$ by setting

$$\text{Hom}_{\mathcal{C}}(f, G(g))(h) = G(g) \circ h \circ f,$$

for every arrow $h : X \longrightarrow GY$ of $\mathcal{C}$.



Proposition 2.1.1. $\mathrm{Hom}_{\mathcal{D}}(F(\cdot),\cdot)$ and $\mathrm{Hom}_{\mathcal{C}}(\cdot,G(\cdot))$ are functors from $\mathcal{C}^{\mathrm{op}}\times\mathcal{D}$ to **Set**.

Proof: We only show that $\mathrm{Hom}_{\mathcal{D}}(F(\cdot),\cdot)$ is a functor. Let $f_1\in\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X_1,X_2)$, $f_2\in\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X_2,X_3)$, $g_1\in\mathrm{Hom}_{\mathcal{D}}(Y_1,Y_2)$ and $g_2\in\mathrm{Hom}_{\mathcal{D}}(Y_2,Y_3)$. Consider $h: FX_1\longrightarrow Y_1$ an arrow of $\mathcal{D}$. We have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(F(f_2\circ f_1),g_2\circ g_1)(h) &= (g_2\circ g_1)\circ h\circ F(f_2\circ f_1)\\ &= (g_2\circ g_1)\circ h\circ (F(f_1)\circ F(f_2))\\ &= g_2\circ (g_1\circ h\circ F(f_1))\circ F(f_2)\\ &= g_2\circ \mathrm{Hom}_{\mathcal{D}}(F(f_1),g_1)(h)\circ F(f_2)\\ &= \mathrm{Hom}_{\mathcal{D}}(F(f_2),g_2)(\mathrm{Hom}_{\mathcal{D}}(F(f_1),g_1)(h)).\end{aligned}$$

Hence $\mathrm{Hom}_{\mathcal{D}}(F(f_2\circ f_1),g_2\circ g_1)=\mathrm{Hom}_{\mathcal{D}}(F(f_2),g_2)\circ\mathrm{Hom}_{\mathcal{D}}(F(f_1),g_1)$. Now let $(\mathrm{id}_X,\mathrm{id}_Y):(X,Y)\longrightarrow(X,Y)$ be the identity arrow of the object (X,Y) of $\mathcal{C}^{\mathrm{op}}\times\mathcal{D}$. Let $h: FX\longrightarrow Y$ be an arrow of $\mathcal{D}$. We have

$$\mathrm{Hom}_{\mathcal{D}}(F(\mathrm{id}_X),\mathrm{id}_Y)(h)=\mathrm{id}_Y\circ h\circ F(\mathrm{id}_X)=h\circ\mathrm{id}_{FX}=h.$$

So we get $\mathrm{Hom}_{\mathcal{D}}(F(\mathrm{id}_X),\mathrm{id}_Y)=\mathrm{id}_{\mathrm{Hom}_{\mathcal{D}}(FX,Y)}$. □

An **adjunction** from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F,G,θ), where $F:\mathcal{C}\longrightarrow\mathcal{D}$ and $G:\mathcal{D}\longrightarrow\mathcal{C}$ are functors, and

$$\theta:\mathrm{Hom}_{\mathcal{D}}(F(\cdot),\cdot)\longrightarrow\mathrm{Hom}_{\mathcal{C}}(\cdot,G(\cdot))$$

is a natural isomorphism. The functor F is said to be the **left adjoint** of G, and G the **right adjoint** of F.

Example 2.1.1. Consider the categories **Set** and $\mathbf{Vct}_{\mathbb{K}}$. Let $U:\mathbf{Vct}_{\mathbb{K}}\longrightarrow\mathbf{Set}$ be the forgetful functor. Define a functor $V:\mathbf{Set}\longrightarrow\mathbf{Vct}_{\mathbb{K}}$ as follows:

(1) $V(X)$ is the vector space with basis X, i.e., the vectors of $V(X)$ are formal finite linear combinations with scalar coefficients;

(2) if $f: X\longrightarrow Y$ is a function, then $V(f):V(X)\longrightarrow V(Y)$ is the map given by $V(f)(x)=f(x)$ at each basis element, and extended by linearity.

Each function $g: X\longrightarrow U(W)$ extends to a unique linear map $f:V(X)\longrightarrow W$, given by $f(\sum r_ix_i)=\sum r_i(g(x_i))$. This correspondence $g\mapsto f$ has an inverse $\theta: f\mapsto f|_X$, the restriction of f to X. Hence, we get a bijection

$$\theta_{X,W}:\mathbf{Vct}_{\mathbb{K}}(V(X),W)\xrightarrow{\cong}\mathbf{Set}(X,U(W)).$$

It is easy to check that θ defines a natural isomorphism. Therefore, (V, U, θ) is an adjunction from **Set** to $\mathbf{Vct}_{\mathbb{K}}$.

For every pair $(X, Y) \in \mathrm{Ob}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D})$, we have a function

$$\theta_{XY} : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, GY).$$

Setting $Y = FX$ we get the following arrow of $\mathcal{C}$:

$$\eta_X := \theta_{X,FX}(\mathrm{id}_{FX}) : X \longrightarrow GFX.$$

Similarly, if we set $X = GY$ then we get the following arrow of $\mathcal{D}$:

$$\epsilon_Y := \theta^{-1}_{GY,Y}(\mathrm{id}_{GY}) : FGY \longrightarrow Y$$

Proposition 2.1.2. $\eta : \mathrm{Id}_{\mathcal{C}} \longrightarrow GF$ and $\epsilon : FG \longrightarrow \mathrm{Id}_{\mathcal{D}}$ define natural transformations.

Proof: We only give a proof for η. A similar argument can be used for ϵ. Given $f : X \longrightarrow X'$ an arrow of $\mathcal{C}$, we must verify that the square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GFX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GF(f)} \\ X' & \xrightarrow[\eta_{X'}]{} & GFX' \end{array}$$

commutes. Consider the arrow $(\mathrm{id}_X, F(f)) : (X, FX) \longrightarrow (X, FX')$ of $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$. Then we have functions

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_{FX}, F(f)) : \mathrm{Hom}_{\mathcal{D}}(FX, FX) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FX, FX')$$

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, GF(f)) : \mathrm{Hom}_{\mathcal{C}}(X, GFX) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, GFX').$$

Since θ is a natural transformation, we have the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX, FX) & \xrightarrow{\theta_{X,FX}} & \mathrm{Hom}_{\mathcal{C}}(X, GFX) \\ {\scriptstyle \mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_{FX}, F(f))}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, GF(f))} \\ \mathrm{Hom}_{\mathcal{D}}(FX, FX') & \xrightarrow[\theta_{X,FX'}]{} & \mathrm{Hom}_{\mathcal{C}}(X, GFX') \end{array}$$

It follows

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, GF(f)) \circ \theta_{X,FX}(\mathrm{id}_{FX}) &= \theta_{X,FX'} \circ \mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_{FX}, F(f))(\mathrm{id}_{FX}) \\ GF(f) \circ \eta_X \circ \mathrm{id}_X &= \theta_{X,FX'}(F(f) \circ \mathrm{id}_{FX} \circ \mathrm{id}_{FX}) \\ GF(f) \circ \eta_X &= \theta_{X,FX'}(F(f)). \end{aligned}$$

Now consider the arrow $(f, \mathrm{id}_{FX'}) : (X', FX') \longrightarrow (X, FX')$ of $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$. Then we have the commutative square

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(FX', FX') & \xrightarrow{\theta_{X',FX'}} & \mathrm{Hom}_{\mathcal{C}}(X', GFX') \\
\downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{D}}(F(f),\mathrm{id}_{FX'})} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f,\mathrm{id}_{GFX'})} \\
\mathrm{Hom}_{\mathcal{D}}(FX, FX') & \xrightarrow[\theta_{X,FX'}]{} & \mathrm{Hom}_{\mathcal{C}}(X, GFX')
\end{array}$$

It follows

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{C}}(f, \mathrm{id}_{GFX'}) \circ \theta_{X',FX'}(\mathrm{id}_{FX'}) & = & \theta_{X,FX'} \circ \mathrm{Hom}_{\mathcal{D}}(F(f), \mathrm{id}_{FX'})(\mathrm{id}_{FX'}) \\
\mathrm{id}_{GFX'} \circ \eta_{X'} \circ f & = & \theta_{X,FX'}(\mathrm{id}_{FX'} \circ \mathrm{id}_{FX'} \circ F(f)) \\
\eta_{X'} \circ f & = & \theta_{X,FX'}(F(f)).
\end{array}$$

Hence $GF(f) \circ \eta_X = \theta_{X,FX'}(F(f)) = \eta_{X'} \circ f$. □

We can say more about η and ϵ. In fact, they are universal arrows. Before stating this fact in detail, we need to prove first the following:

Lemma 2.1.1. For a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$, a pair $(D, \mu : C \longrightarrow G(D))$ is a universal arrow from C to G if and only if the function sending each arrow $f : D \longrightarrow D'$ of $\mathcal{D}$ into $G(f) \circ \mu : C \longrightarrow G(D')$ is a bijection from $\mathrm{Hom}_{\mathcal{D}}(D, D')$ to $\mathrm{Hom}_{\mathcal{C}}(C, G(D'))$.

Proof: The function $\varphi : \mathrm{Hom}_{\mathcal{D}}(D, D') \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, G(D'))$ given by $\varphi(f) = G(f) \circ \mu$ is a bijection if and only if for every arrow $g : C \longrightarrow G(D')$ of $\mathcal{C}$ there exists a unique arrow $f : D \longrightarrow D'$ of $\mathcal{D}$ such that $g = \varphi(f)$, i.e., if and only if the triangle

$$\begin{array}{ccc}
C & \xrightarrow{\mu} & G(D) \\
& {\scriptstyle g}\searrow & \downarrow{\scriptstyle G(f)} \\
& & G(D')
\end{array}$$

commutes for one and only one arrow $f : D \longrightarrow D'$ of $\mathcal{D}$. □

Proposition 2.1.3.

(1) For each $C \in \mathrm{Ob}(\mathcal{C})$, $\eta_C : C \longrightarrow GF(C)$ is a universal arrow from C to G.

(2) For each $D \in \mathrm{Ob}(\mathcal{D})$, $\epsilon_D : FG(D) \longrightarrow D$ is a universal arrow from F to D.

Proof: We only prove (1). By the previous lemma, we only need to show that the function $\varphi : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ given by $\varphi(f) = G(f) \circ \eta_C$ is a bijection. Let $f : F(C) \longrightarrow D$ be an arrow of $\mathcal{D}$. Since the square

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{D}}(FC, FC) & \xrightarrow{\theta_{C,FC}} & \mathrm{Hom}_{\mathcal{C}}(C, GFC) \\
{\scriptstyle \mathrm{Hom}_{\mathcal{D}}(F(f), \mathrm{id}_{FC})} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_C, G(f))} \\
\mathrm{Hom}_{\mathcal{D}}(FC, D) & \xrightarrow[\theta_{C,D}]{} & \mathrm{Hom}_{\mathcal{C}}(C, GD)
\end{array}$$

commutes, we have that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_C, G(f)) \circ \theta_{C,FC}(\mathrm{id}_{FC}) &= \theta_{C,D} \circ \mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_{FC}, f)(\mathrm{id}_{FC}) \\
G(f) \circ \eta_C \circ \mathrm{id}_C &= \theta_{C,D}(f \circ \mathrm{id}_{FC} \circ \mathrm{id}_{FC}) \\
G(f) \circ \eta_C &= \theta_{C,D}(f) \\
\varphi(f) &= \theta_{C,D}(f).
\end{aligned}$$

Notice that θ is a natural isomorphism and hence $\varphi = \theta_{C,D}$ is a bijection. $\square$

Theorem 2.1.1 (Pointwise Construction of Adjoints)**.**

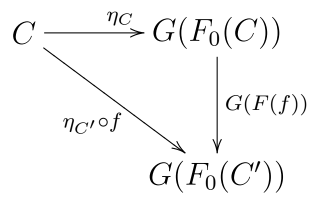
(1) A functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint if and only if for each $C \in \mathrm{Ob}(\mathcal{C})$ there is an object $F_0(C)$ of $\mathcal{D}$ and a universal arrow $\eta_C : C \longrightarrow G(F_0(C))$ from C to G.

(2) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has a right adjoint if and only if for each $D \in \mathrm{Ob}(\mathcal{D})$ there is an object $G_0(D)$ of $\mathcal{C}$ and a universal arrow $\epsilon_D : F(G_0(D)) \longrightarrow D$ from F to D.

Proof: We only prove (1). If (F, G, θ) is an adjunction then we can set $F_0 = F$ and $\eta_C = \theta_{C,FC}(\mathrm{id}_{FC})$. By the previous proposition, η_C is a universal arrow from C to G, for each $C \in \mathrm{Ob}(\mathcal{C})$. Now suppose that for each object C of $\mathcal{C}$ there exist an object $F_0(C)$ of $\mathcal{D}$ and a universal arrow $\eta_C : C \longrightarrow G(F_0(C))$ from C to G. We define a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ by setting

(1) $F(C) = F_0(C)$ for each $C \in \mathrm{Ob}(\mathcal{C})$, and

(2) if $f : C \longrightarrow C'$ is an arrow of $\mathcal{C}$ then $F(f)$ is the only arrow of $\mathcal{D}$ such that the triangle



commutes.

Let $f : C \longrightarrow C'$ and $g : C' \longrightarrow C''$ be two arrows of $\mathcal{C}$. We know that $F(g \circ f) : F(C) \longrightarrow F(C'')$ is the only arrow of $\mathcal{D}$ such that $G(F(g \circ f)) \circ \eta_C = \eta_{C''} \circ (g \circ f)$. On the other hand,

$$\begin{aligned} G(F(g) \circ F(f)) \circ \eta_C &= [G(F(g)) \circ G(F(f))] \circ \eta_C = G(F(g)) \circ [G(F(f)) \circ \eta_C] \\ &= G(F(g)) \circ [\eta_{C'} \circ f] = [GF(g) \circ \eta_{C'}] \circ f \\ &= [\eta_{C''} \circ g] \circ f = \eta_{C''} \circ (g \circ f). \end{aligned}$$

It follows $F(g \circ f) = F(g) \circ F(f)$. Similarly, one can prove that $F(\mathrm{id}_C) = \mathrm{id}_{FC}$.

It is only left to construct a natural isomorphism

$$\theta : \mathrm{Hom}_{\mathcal{D}}(F(\cdot), \cdot) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\cdot, G(\cdot)).$$

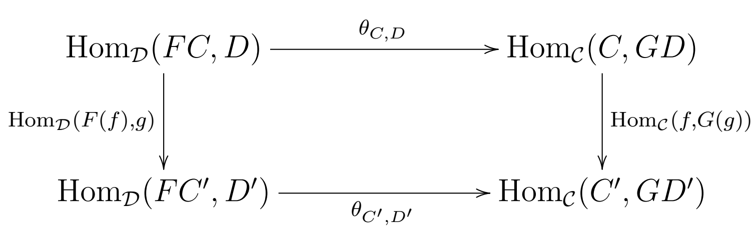
Let $C \in \mathrm{Ob}(\mathcal{C})$ and $D \in \mathrm{Ob}(\mathcal{D})$. We define a function

$$\theta_{C,D} : \mathrm{Hom}_{\mathcal{D}}(FC, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, GD)$$

by $\theta_{C,D}(h) = G(h) \circ \eta_C$, for every arrow $h : FC \longrightarrow D$ of $\mathcal{D}$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(f, G(g)) \circ \theta_{C,D}(h) &= \mathrm{Hom}_{\mathcal{C}}(f, G(g))(G(h) \circ \eta_C) = G(g) \circ (G(h) \circ \eta_C) \circ f \\ &= (G(g) \circ G(h)) \circ \eta_C \circ f = G(g \circ h) \circ \eta_C \circ f, \\ \theta_{C',D'} \circ \mathrm{Hom}_{\mathcal{D}}(F(f), g)(h) &= \theta_{C',D'}(g \circ h \circ F(f)) = G(g \circ h \circ F(f)) \circ \eta_{C'} \\ &= G(g \circ h) \circ G(F(f)) \circ \eta_{C'} = G(g \circ h) \circ \eta_C \circ f. \end{aligned}$$

Hence, we get the following commutative square



By Lemma 2.1.1, each $\theta_{C,D}$ is bijective. Therefore, θ is a natural isomorphism. □

2.2 Preservation of limits

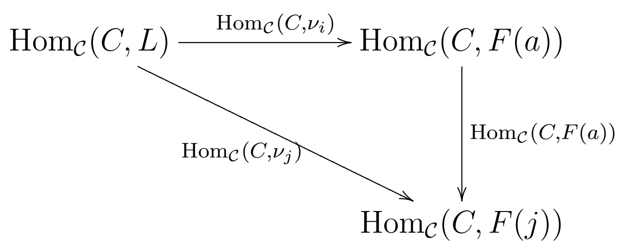
A functor $H : \mathcal{C} \longrightarrow \mathcal{D}$ is said to **preserve** limits of functors $F : \mathcal{J} \longrightarrow \mathcal{C}$ when if $(L, (\nu_i : L \longrightarrow F(i))_{i \in \mathrm{Ob}(\mathcal{J})})$ is the limit for F, then $(H(L), (H(\nu_i) : H(L) \longrightarrow HF(i))_{i \in \mathrm{Ob}(\mathcal{J})})$ is the limit for HF.

Example 2.2.1. If $\mathcal{C}$ is a small category, then the functor $\mathrm{Hom}_{\mathcal{C}}(C, \cdot)$ preserves limits.

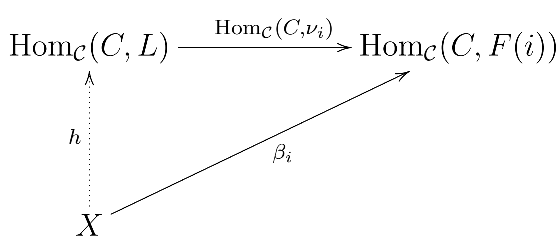
Suppose that $(L, (\nu_i : L \longrightarrow F(i))_{i \in \mathrm{Ob}(\mathcal{J})})$ is the limit for a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$. Consider the family of functions $(\mathrm{Hom}_{\mathcal{C}}(C, \nu_i))_{i \in \mathrm{Ob}(\mathcal{J})}$. First, if $a : i \longrightarrow j$ is an arrow of $\mathcal{J}$ and $f : C \longrightarrow L$ is an arrow of $\mathcal{C}$, then we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(C, F(a)) \circ \mathrm{Hom}_{\mathcal{C}}(C, \nu_i)(f) = \mathrm{Hom}_{\mathcal{C}}(C, F(a))(\nu_i \circ f) = F(a) \circ (\nu_i \circ f) = (F(a) \circ \nu_i) \circ f \\ = \nu_j \circ f = \mathrm{Hom}_{\mathcal{C}}(C, \nu_j)(f),\end{aligned}$$

i.e., the triangle



commutes. Now let $(\beta_i : X \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, F(i)))_{i \in \mathrm{Ob}(\mathcal{J})}$ be another family of functions such that $\mathrm{Hom}_{\mathcal{C}}(C, F(a)) \circ \beta_i = \beta_j$, for every arrow $a : i \longrightarrow j$. Then for each $x \in X$, we have $\beta_j(x) = F(a) \circ \beta_i(x)$, where each $\beta_i(x) : C \longrightarrow F(i)$ is an arrow of $\mathcal{C}$. Since $(\nu_i : L \longrightarrow F(i))_{i \in \mathrm{Ob}(\mathcal{J})}$ is the limit for F, then there exists a unique arrow $h_x : C \longrightarrow L$ of $\mathcal{C}$ such that $\beta_i(x) = \nu_i \circ h_x$. Thus, we get a well defined function $h : X \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, L)$ given by $h(x) = h_x$, for each $x \in X$. Clearly, we have that the commutative triangle



It is only left to show that h is the only function satisfying $\mathrm{Hom}_{\mathcal{C}}(C, \nu_i) \circ h = \beta_i$. Let $h' : X \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, L)$ be another function such that $\mathrm{Hom}_{\mathcal{C}}(C, \nu_i) \circ h' = \beta_i$. Then for each $x \in X$, we have $\nu_i \circ h'(x) = \beta_i(x)$. On the other hand, h_x is the only arrow of $\mathcal{C}$ satisfying $\nu_i \circ h_x = \beta_i(x)$. So we get $h_x = h'(x)$, for each $x \in X$, i.e., $h = h'$.

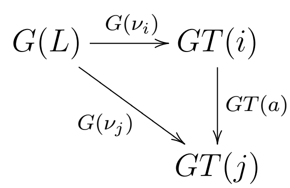
Theorem 2.2.1. If (F, G, θ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$, then G preserves limits and F preserves colimits.

Proof: We prove that G preserves limits. Let $(L, (\nu_i : L \longrightarrow T(i))_{i \in \mathrm{Ob}(\mathcal{J})})$ be the limit for a functor $T : \mathcal{J} \longrightarrow \mathcal{D}$. Consider the family of arrows $(G(\nu_i) : GL \longrightarrow GT(i))_{i \in \mathrm{Ob}(\mathcal{J})}$ of $\mathcal{C}$.

(1) If $a : i \longrightarrow j$ is an arrow of $\mathcal{J}$, then we have

$$GT(a) \circ G(\nu_i) = G(T(a) \circ \nu_i) = G(\nu_j),$$

since G is a functor and $(\nu_i : L \longrightarrow T(i))_{i \in \mathrm{Ob}(\mathcal{J})}$ is the limit for T. Hence, the triangle

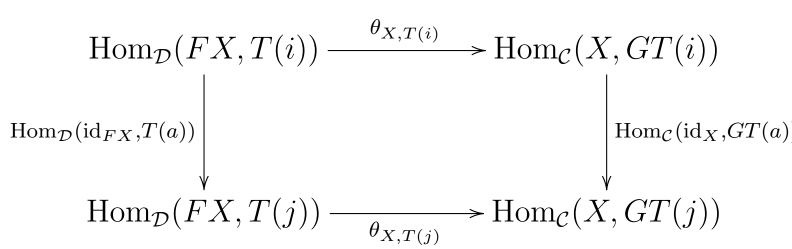


commutes.

(2) Let $(f_i : X \longrightarrow GT(i))_{i \in \mathrm{Ob}(\mathcal{J})}$ be another family of arrows of $\mathcal{C}$ with $GT(a) \circ f_i = f_j$. Recall we have a natural isomorphism

$$\theta : \mathrm{Hom}_{\mathcal{D}}(F(\cdot), \cdot) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\cdot, G(\cdot)).$$

Note that $f_i \in \mathrm{Hom}_{\mathcal{C}}(X, GT(i))$. Thus, set $g_i = \theta^{-1}_{X,T(i)}(f_i) \in \mathrm{Hom}_{\mathcal{D}}(FX, T(i))$. Consider the commutative square



where $a : i \longrightarrow j$ is an arrow of $\mathcal{J}$. We get

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_{FX}, T(a)) \circ \theta^{-1}_{X,T(i)}(f_i) &= \theta^{-1}_{X,T(j)} \circ \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, GT(a))(f_i) \\
T(a) \circ \theta^{-1}_{X,T(i)}(f_i) \circ \mathrm{id}_{FX} &= \theta^{-1}_{X,T(j)}(GT(a) \circ f_i \circ \mathrm{id}_X) \\
T(a) \circ \theta^{-1}_{X,T(i)}(f_i) &= \theta^{-1}_{X,T(j)}(f_j) \\
T(a) \circ g_i &= g_j.
\end{aligned}$$

Hence, we obtain the following commutative triangle in $\mathcal{D}$:

$$\begin{array}{ccc} FX & \xrightarrow{g_i} & T(i) \\ & \searrow^{g_j} & \downarrow{\scriptstyle T(a)} \\ & & T(j) \end{array}$$

It follows there exists a unique arrow of $\mathcal{D}$, say $g : FX \longrightarrow L$, making the following triangle commute, for each $i \in \mathrm{Ob}(\mathcal{J})$:

$$\begin{array}{ccc} L & \xrightarrow{\nu_i} & T(i) \\ {\scriptstyle g}\uparrow & \nearrow_{g_i} & \\ FX & & \end{array}$$

Now set $h = \theta_{X,L}(g) \in \mathrm{Hom}(X, GL)$. Since the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX, L) & \xrightarrow{\theta_{X,L}} & \mathrm{Hom}_{\mathcal{C}}(X, GL) \\ {\scriptstyle \mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_{FX}, \nu_i)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, G(\nu_i))} \\ \mathrm{Hom}_{\mathcal{D}}(FX, T(i)) & \xrightarrow[\theta_{X,T(i)}]{} & \mathrm{Hom}_{\mathcal{C}}(X, GT(i)) \end{array}$$

commutes, it follows that

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, G(\nu_i)) \circ \theta_{X,L}(g) & = & \theta_{X,T(i)} \circ \mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_{FX}, \nu_i)(g) \\ G(\nu_i) \circ \theta_{X,L}(g) \circ \mathrm{id}_X & = & \theta_{X,T(i)}(\nu_i \circ g \circ \mathrm{id}_{FX}) \\ G(\nu_i) \circ h & = & \theta_{X,T(i)}(g_i) \\ G(\nu_i) \circ h & = & f_i, \end{array}$$

i.e., the triangle

$$\begin{array}{ccc} GL & \xrightarrow{G(\nu_i)} & GT(i) \\ {\scriptstyle h}\uparrow & \nearrow_{f_i} & \\ X & & \end{array}$$

commutes. It is only left to show that $h : X \longrightarrow GL$ is the only arrow of $\mathcal{C}$ satisfying $G(\nu_i) \circ h = f_i$. Suppose there is another arrow of $h' : X \longrightarrow GL$ of $\mathcal{C}$ satisfying $G(\nu_i) \circ h' = f_i$. Consider the arrow $\theta^{-1}_{X,L}(h') \in \mathrm{Hom}_{\mathcal{D}}(FX, L)$. Since the above square commutes, we get

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{D}}(\mathrm{id}_{FX}, \nu_i) \circ \theta^{-1}_{X,L}(h') & = & \theta^{-1}_{X,T(i)} \circ \mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_X, G(\nu_i))(h') \\ \nu_i \circ \theta^{-1}_{X,L}(h') & = & \theta^{-1}_{X,T(i)}(G(\nu_i) \circ h') \\ \nu_i \circ \theta^{-1}_{X,L}(h') & = & g_i. \end{array}$$

On the other hand, g is the only arrow of $\mathcal{D}$ satisfying $\nu_i \circ g = g_i$. Hence, we have $g = \theta^{-1}_{X,L}(h')$, i.e., $h' = \theta_{X,L}(g) = h$.

Therefore, $(GL, (G(\nu_i) : GL \longrightarrow GT(i))_{i \in \mathrm{Ob}(\mathcal{J})})$ is the limit for the functor GT. $\square$

2.3 The Freyd's Adjoint Functor Theroem

Lemma 2.3.1 (Existence of an initial object)**.** Let $\mathcal{D}$ be a small and complete category. Then $\mathcal{D}$ has an initial object if and only if it satisfies the following
Solution Set Condition: There exists a set I and an i-indexed family $(D_i)_{i\in I}$ of objects of $\mathcal{D}$ such that for every $D \in \mathrm{Ob}(\mathcal{D})$ there is an $i \in I$ and an arrow $D_i \longrightarrow D$ of $\mathcal{D}$.

Proof: If $\mathcal{D}$ has an initial object D_0 then set $I = \{0\}$. Then for every $D \in \mathrm{Ob}(\mathcal{D})$ there exists one and only one arrow $D_0 \longrightarrow D$, i.e., $\{D_0\}$ satisfies the Solution Set Condition. Now suppose that $\mathcal{D}$ satisfies the Solution Set Condition. Let $(P, (p_i : P \longrightarrow D_i)_{i\in I})$ be the product of the family $(D_i)_{i\in I}$. Let $\mathcal{J}$ be the small category whose only object is P and $\mathrm{Hom}_{\mathcal{J}}(P,P) = \mathrm{Hom}_{\mathcal{D}}(P,P)$. Define a functor $F : \mathcal{J} \longrightarrow \mathcal{D}$ by $F(P) = P$ and $F(f) = f$, for every $f \in \mathrm{Hom}_{\mathcal{J}}(P,P)$. Let $(L, (\nu : L \longrightarrow P))$ be the limit for F. Then the triangle

$$\begin{array}{ccc} L & \xrightarrow{\nu} & P \\ & \searrow^{\nu} & \downarrow{\scriptstyle F(f)} \\ & & P \end{array}$$

commutes for every $f \in \mathrm{Hom}_{\mathcal{J}}(P,P)$. It follows that $f \circ \nu = g \circ \nu$, for every $f, g : P \longrightarrow P$ in $\mathcal{D}$. Let $\nu' : L' \longrightarrow P$ be another arrow of $\mathcal{D}$ such that $f \circ \nu' = \nu'$ for every f. Then there exists a unique arrow $h : L' \longrightarrow L$ of $\mathcal{D}$ such that $\nu' = \nu \circ h$; i.e., if $\nu' : L' \longrightarrow P$ is another arrow such that $f \circ \nu' = g \circ \nu'$ for every $f, g \in \mathrm{Hom}_{\mathcal{D}}(P,P)$, then ν' factors uniquely through ν. Hence, $\nu : L \longrightarrow P$ is the equalizer of the set $\mathrm{Hom}_{\mathcal{D}}(P,P)$. We shall see that L is an initial object in $\mathcal{D}$. Let $D \in \mathrm{Ob}(\mathcal{D})$. There is an $i \in I$ and an arrow $f_i : D_i \longrightarrow D$. Then $f_i \circ p_i \circ \nu$ is an arrow from L to D. Now we prove that such an arrow is unique. Let $f, g : L \longrightarrow D$ be two arrows of $\mathcal{D}$ and $\mu : K \longrightarrow L$ the equalizer of f and g. There is an $i \in I$ and an arrow $D_i \longrightarrow K$. Then $s : P \xrightarrow{p_i} D_i \longrightarrow K$ is an arrow from P to K. So we get the arrow $P \xrightarrow{s} K \xrightarrow{\mu} L \xrightarrow{\nu} P$ in $\mathrm{Hom}_{\mathcal{D}}(P,P)$. Since ν is the equalizer of $\mathrm{Hom}_{\mathcal{D}}(P,P)$, we have that $(\nu \circ \mu \circ s) \circ \nu = \mathrm{id}_P \circ \nu$. It follows $\nu \circ (\mu \circ s \circ \nu) = \nu \circ \mathrm{id}_L$. Hence $\mu \circ s \circ \nu = \mathrm{id}_L$ since equalizers are monic. We have that μ is a monic arrow having a right inverse. It follows μ is an isomorphism. Hence $f \circ \mu = g \circ \mu$ implies that $f = g$. $\square$

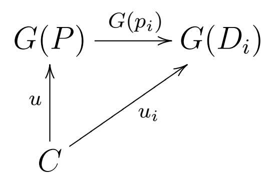
Lemma 2.3.2. If $\mathcal{D}$ is a complete category and $G : \mathcal{D} \longrightarrow \mathcal{C}$ is a functor that preserves products and equalizers, then the comma category $(C \downarrow G)$ is complete, for each $C \in \mathrm{Ob}(\mathcal{C})$.

Proof: By Theorem 1.2.1 it is only left to prove that $(C \downarrow G)$ has products and equalizers.

(1) $(C \downarrow G)$ has products: Let $(u_i, D_i)_{i\in I}$ be a family of objects in $(C \downarrow G)$. Since $\mathcal{D}$ is complete, there exists the product of the family $(D_i)_{i\in I}$, say $(P, (p_i : P \longrightarrow D_i)_{i\in I})$. On the other hand, G preserves products, so

$$(G(P), (G(p_i) : G(P) \longrightarrow G(D_i))_{i\in I})$$

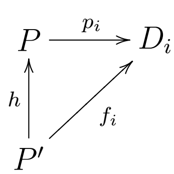
is the product of $(G(D_i))_{i\in I}$. Since the arrows $G(p_i)$ are universal, there exists a unique arrow $u : C \longrightarrow G(P)$ such that the triangle



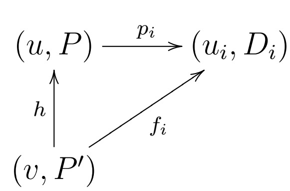
commutes. Thus, (u, P) is an object of $(C \downarrow G)$ and $p_i : (u, P) \longrightarrow (u_i, D_i)$ is an arrow of $(C \downarrow G)$. We show that (u, P) is the product of $(u_i, D_i)_{i\in I}$. Let

$$(f_i : (v, P') \longrightarrow (u_i, D_i))_{i\in I}$$

be another family of arrows in $(C \downarrow G)$. Since the arrows p_i are universal in $\mathcal{D}$, there exists a unique arrow $h : P' \longrightarrow P$ such that the triangle

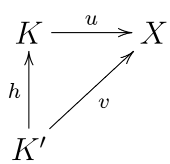


commutes. Recall that u is the only arrow of $\mathcal{C}$ satisfying $G(p_i) \circ u = u_i$. Also, $G(p_i)\circ G(h)\circ v = G(p_i\circ h)\circ v = G(f_i)\circ v = u_i$, for every $i \in I$. It follows $u = G(h)\circ v$ and so $h : (v, P') \longrightarrow (u, P)$ is an arrow of $(C \downarrow G)$ such that $p_i \circ h = f_i$, i.e., the triangle

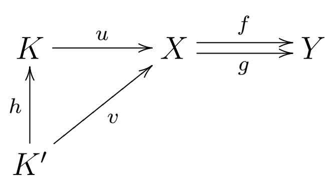


commutes. Now suppose that there exists another arrow $h' : (v, P') \longrightarrow (u, P)$ of $(C \downarrow G)$ such that $p_i \circ h' = f_i$. Since h is the only arrow of $\mathcal{D}$ satisfying $p_i \circ h = f_i$, we have $h' = h$.

(2) $(C \downarrow G)$ has equalizers: Let $f, g : (s, X) \longrightarrow (r, S)$ be two parallel arrows on $(C \downarrow G)$. Then $G(f) \circ s = r = G(g) \circ s$. Since $\mathcal{D}$ is complete, there exists the equalizer of $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ as arrows of $\mathcal{D}$, say $u : K \longrightarrow X$. But G preserves equalizers, then $G(u)$ is the equalizer of $G(f)$ and $G(g)$. Since $G(f) \circ s = G(g) \circ s$, there exists a unique arrow $\lambda : C \longrightarrow G(K)$ of $\mathcal{C}$ such that $G(u) \circ \lambda = s$. Thus, $u : (\lambda, K) \longrightarrow (s, X)$ is an arrow of $(C \downarrow G)$. We prove that u is the equalizer of f and g, as arrows of $(C \downarrow G)$. Let $v : (\lambda', K') \longrightarrow (s, X)$ be another arrow of $(C \downarrow G)$ such that $f \circ v = g \circ v$. Since $u : K \longrightarrow X$ is the equalizer of f and g as arrows of $\mathcal{D}$, there exists a unique arrow $h : K \longrightarrow K$ such that the triangle



commutes. Then we have $G(u) \circ (G(h) \circ \lambda') = G(u \circ h) \circ \lambda' = G(v) \circ \lambda' = s$. On the other hand, λ is the only arrow of $\mathcal{C}$ satisfying $G(u) \circ \lambda = s$. It follows $G(h) \circ \lambda' = \lambda$, i.e., $h : (\lambda', K') \longrightarrow (\lambda, K)$ is an arrow of $(C \downarrow G)$. If $h' : (\lambda', K') \longrightarrow (\lambda, K)$ is another arrow satisfying $u \circ h' = v$, then $h' = h$ since h is the only arrow of $\mathcal{D}$ making the diagram



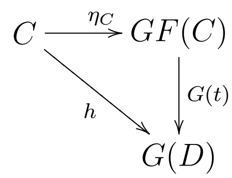
commute.

□

Theorem 2.3.1 (Freyd)**.** Given a small and complete category $\mathcal{D}$, a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint if and only if it preserves all limits and satisfies the following
Solution Set Condition. For each object $C \in \mathrm{Ob}(\mathcal{C})$ there is a set I and an I-indexed family of arrows $f_i : C \longrightarrow G(D_i)$ such that every arrow $h : C \longrightarrow G(D)$ can be written as a composite $h = G(t) \circ f_i$ for some index i and some $t : D_i \longrightarrow D$.

Proof: Suppose $G : \mathcal{D} \longrightarrow \mathcal{C}$ has a left adjoint $F : \mathcal{C} \longrightarrow \mathcal{D}$. By Theorem 2.2.1, G preserves all limits in $\mathcal{D}$. Consider the unit $\eta : \mathrm{Id}_{\mathcal{C}} \longrightarrow GF$. For each $C \in \mathrm{Ob}(\mathcal{C})$, we get a universal arrow $\eta_C : C \longrightarrow GF(C)$ (see Proposition 2.1.3). Let $I = \{C\}$, $D_C = F(C)$

and $f_C = \eta_C$. Let $h : C \longrightarrow G(D)$ be an arrow in $\mathcal{C}$ of $\mathcal{C}$. Since η_C is universal, there exists a unique arrow $t : F(C) \longrightarrow D$ of $\mathcal{D}$ such that the triangle



commutes, i.e., $G(t) \circ f_C = h$. Hence, G satisfies the Solution Set Condition. Now suppose that G is a functor which preserves limits and satisfies the Solution Set Condition. By Theorem 2.1.1, we only need to construct a universal arrow from C to G, $\eta_C : C \longrightarrow G(F_0(C))$, for each $C \in \mathrm{Ob}(\mathcal{C})$. By Proposition 1.1.2, such an arrow $(\eta_C, F_0(C))$ is an initial object in the comma category $(C \downarrow G)$. By Lemma 2.3.2, $(C \downarrow G)$ is complete. On the other hand, there exists an I-indexed family of arrows $f_i : C \longrightarrow G(D_i)$ such that every arrow $h : C \longrightarrow G(D)$ can be written as a composition $h = G(t) \circ f_i$ for some $i \in I$ and some arrow $t : D_i \longrightarrow D$ of $\mathcal{D}$. In other words, there exists a set I and an I-indexed family of objects (f_i, D_i) of $(C \downarrow G)$ such that for every object (h, D) of $(C \downarrow G)$ there is an $i \in I$ and an arrow $t : (f_i, D_i) \longrightarrow (h, D)$ of $(C \downarrow G)$. We have that $(C \downarrow G)$ is a small and complete category satisfying the Solution Set Condition given in Lemma 2.3.1. Therefore, $(C \downarrow G)$ has an initial object $(\eta_C, F_0(C))$. □

Using the dual statements of both Lemma 2.3.1 and 2.3.2, on can prove the dual version of the Freyd's theorem.

Theorem 2.3.2 (Freyd)**.** Given a small and complete category $\mathcal{C}$, a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has a right adjoint if and only if it preserves all colimits and satisfies the following
Solution Set Condition. For each object $D \in \mathrm{Ob}(\mathcal{D})$ there is a set I and an I-indexed family of arrows $g_i : F(C_i) \longrightarrow D$ such that every arrow $h : F(C) \longrightarrow D$ can be written as a composite $h = g_i \circ F(t)$ for some index i and some $t : C \longrightarrow C_i$.

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