Tensor Products and Internal Homs for Chain Complexes

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Abstract

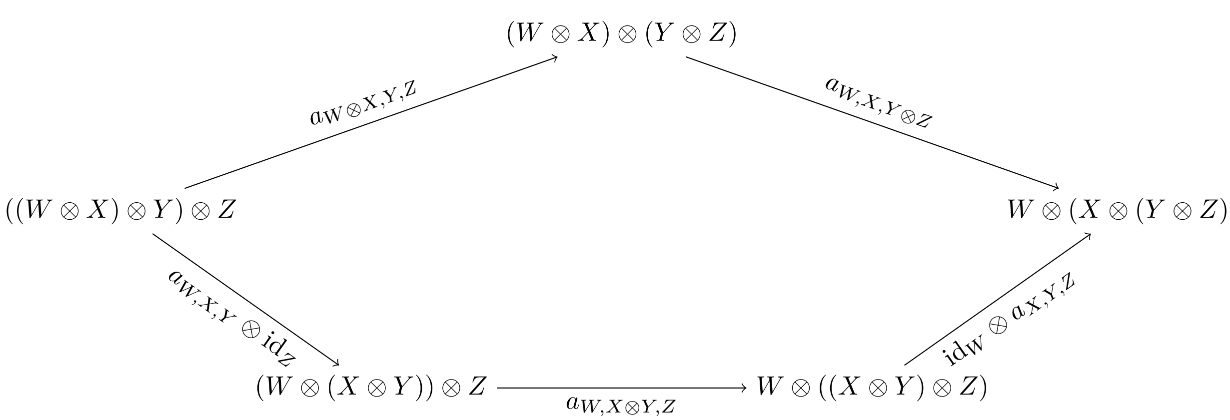
These notes are devoted to study the standard and the bar tensor products, $\otimes$ and $\overline{\otimes}$, defined on the category of chain complexes. We prove that the complex $\mathrm{Hom}'(X,Y)$ is the internal Hom of the monodical category $(\mathbf{Ch}({}_R\mathbf{Mod}), \otimes)$. From $\mathrm{Hom}'(X,Y)$, it is possible to construct a functor $\overline{\mathrm{Hom}}(X,Y)$ which turn out to be the internal Hom of the monodical category $(\mathbf{Ch}({}_R\mathbf{Mod}), \overline{\otimes})$.

Contents

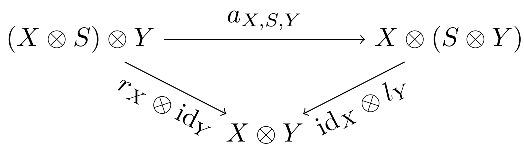
1 Monoidal categories

A **symmetric monoidal structure** on a category $\mathcal{C}$ is given by a tensor product bifunctor $-\otimes- : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, a **unit object** $S \in \mathrm{Ob}(\mathcal{C})$, and natural isomorphisms:

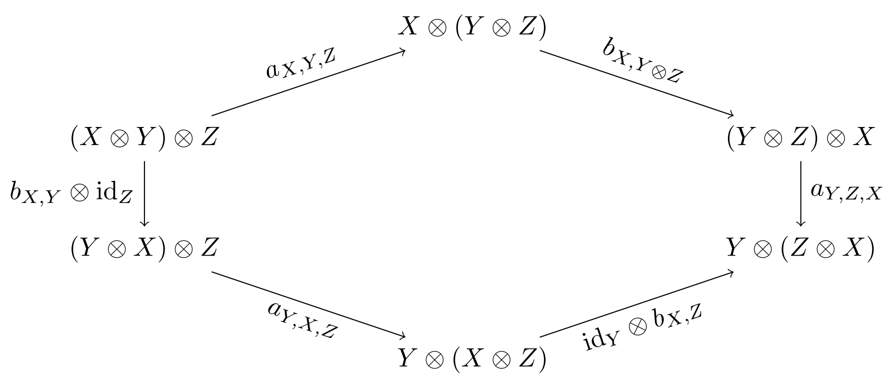
- **Associativity**: $(-\otimes-)\otimes- \xrightarrow{a} -\otimes(-\otimes-)$ where $a_{X,Y,Z} : (X\otimes Y)\otimes Z \mapsto X\otimes(Y\otimes Z)$.
- **Left unit**: $S\otimes- \xrightarrow{l} \mathrm{id}_{\mathcal{C}}$ where $l_Y : S\otimes Y \mapsto Y$.
- **Right unit**: $-\otimes S \xrightarrow{r} \mathrm{id}_{\mathcal{C}}$ where $r_X : X\otimes S \mapsto X$.
- **Braiding**: $-\otimes- \xrightarrow{b} -\otimes^{\mathrm{op}}-$ where $X\otimes^{\mathrm{op}} Y = Y\otimes X$ and $b_{X,Y} : X\otimes Y \to Y\otimes X$, such that the following diagrams, called **coherence diagrams**, commute:
  (1) Pentagon identity:



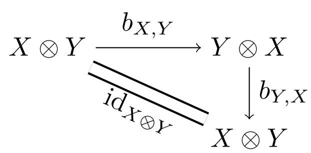
(2) Triangle identity:



(3) Hexagon identity:



(4)



We denote the symmetric monoidal structure on $\mathcal{C}$ by the quintuple $(\otimes, a, b, r, l)$.

The following definitions appear in [2, Sections 4.1 and 4.2], which provides a detailed study of monoidal model categories.

A symmetric monoidal structure $(\otimes, a, b, l, r)$ on $\mathcal{C}$ is said to be **closed** if for every object $Y \in \mathrm{Ob}(\mathcal{C})$ the functor $- \otimes Y : \mathcal{C} \to \mathcal{C}$ has a right adjoint functor $[Y, -] : \mathcal{C} \to \mathcal{C}$. This means that for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ we have a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, [Y, Z])$. Some authors call the right adjoint $[-, -]$ the **internal Hom**.

Example 1.1. Let R be a commutative ring. Then $({}_R\mathbf{Mod}, \otimes_R)$ is a symmetric monoidal category, where $\otimes_R$ is the usual tensor product of modules, and R is the unit object.

2 The standard tensor product of chain complexes

From now on, R shall denote a commutative ring with unit 1. In this section, we prove that the functor $\mathrm{Hom}'(-, -)$ is the internal Hom of the standard tensor product $- \otimes -$. Recall that given two chain complexes $X, Y \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$, the tensor product $X \otimes Y$ is the chain complex given by

$$\boxed{(X \otimes Y)_n := \bigoplus_{k \in \mathbb{Z}} X_k \otimes_R Y_{n-k}}$$

whose boundary maps $\partial_n^{X\otimes Y} : (X\otimes Y)_n \longrightarrow (X\otimes Y)_{n-1}$ are given by

$$\boxed{x\otimes y \mapsto \partial^X(x)\otimes y + (-1)^{|x|} x\otimes \partial^Y(y)}$$

where $|x| = k$ if $x \in X_k$. The category $\mathbf{Ch}({}_R\mathbf{Mod})$ equipped with the tensor product $\otimes$ turns out to be a symmetric monodical category $(\mathbf{Ch}({}_R\mathbf{Mod}), \otimes)$, where the unit object is given by 0th sphere complex $S^0(R)$ defined by

$$\boxed{(S^0(R))_k := \begin{cases} R & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}}$$

where the boundary maps are all zero.

On the other hand, $\mathrm{Hom}'(X, Y)$ is the complex defined by

$$\boxed{\mathrm{Hom}'(X, Y)_n := \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(X_k, Y_{n+k})}$$

where the boundary maps $\partial_n^{\mathrm{Hom}'(X,Y)} : \mathrm{Hom}'(X, Y)_n \longrightarrow \mathrm{Hom}'(X, Y)_{n-1}$ are given by

$$\boxed{(f_k : X_k \longrightarrow Y_{n+k})_{k\in\mathbb{Z}} \mapsto (\partial^Y_{n+k} \circ f_k - (-1)^n f_{k-1} \circ \partial^X_k)_{k\in\mathbb{Z}}}$$

The internal Hom of $(\mathbf{Ch}({}_R\mathbf{Mod}), \otimes)$ is given by $\mathrm{Hom}'_{\mathbf{Ch}({}_R\mathbf{Mod})}(-,-)$, as mentioned in [2, Proposition 4.2.13]. The goal in this section is to prove the following result.

Theorem 2.1

Let R be a commutative ring with unit, and $X, Y, Z \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$. Then there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X, \mathrm{Hom}'(Y, Z)) \cong \mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X\otimes Y, Z)$$

We shall construct an isomorphism of groups

$$\Psi : \mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X, \mathrm{Hom}'(Y, Z)) \longrightarrow \mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X\otimes Y, Z).$$

Consider a chain map $g : X \longrightarrow \mathrm{Hom}'(Y, Z)$. We want $\Psi(g)$ to be a chain map $X\otimes Y \longrightarrow Z$. Then, for each $n \in \mathbb{Z}$, we need $\Psi(g)_n : \bigoplus_{k\in\mathbb{Z}} X_k \otimes_R Y_{n-k} \longrightarrow Z_n$.

Let $x\otimes y \in X_k \otimes_R Y_{n-k}$. Then $g_k(x) \in \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(Y_i, Z_{k+i})$. It follows $g_k(x)_{n-k}(y) \in Z_n$. So define

$$\boxed{\Psi(g)_n(x\otimes y) := g_k(x)_{n-k}(y) \text{ for every } x\otimes y \in X_k \otimes_R Y_{n-k}}$$

In order to show that $\Psi(g) = (\Psi(g)_n)_{n\in\mathbb{Z}}$ defines a chain map, we need to check that the following squares commutes:

$$\begin{array}{ccc} (X\otimes Y)_n & \xrightarrow{\Psi(g)_n} & Z_n \\ {\scriptstyle \partial_n^{X\otimes Y}} \downarrow & & \downarrow {\scriptstyle \partial_n^Z} \\ (X\otimes Y)_{n-1} & \xrightarrow[\Psi(g)_{n-1}]{} & Z_{n-1} \end{array}$$

We have:

$$\begin{aligned}\Psi(g)_{n-1} \circ \partial_n^{X\otimes Y}(x \otimes y) &= \Psi(g)_{n-1}(\partial_k^X(x) \otimes y + (-1)^k x \otimes \partial_{n-k}^Y(y)) \\ &= \Psi(g)_{n-1}(\partial_k^X(x) \otimes y) + (-1)^k \Psi(g)_{n-1}(x \otimes \partial_{n-k}^Y(y)) \\ &= g_{k-1}(\partial_k^X(x))_{n-k}(y) + (-1)^k g_k(x)_{n-k-1}(\partial_{n-k}^Y(y)).\end{aligned}$$

On the other hand, the following square commutes since g is a chain map:

$$\begin{array}{ccc} X_k & \xrightarrow{\quad g_k \quad} & \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(Y_i, Z_{k+i}) \\ {\scriptstyle \partial_k^X}\big\downarrow & & \big\downarrow{\scriptstyle \partial_k^{\operatorname{Hom}'(Y,Z)}} \\ X_{k-1} & \xrightarrow[\quad g_{k-1} \quad]{} & \prod_{i\in\mathbb{Z}} \operatorname{Hom}_R(Y_i, Z_{k-1+i}) \end{array}$$

Then we have:

$$\begin{aligned} g_{k-1}(\partial_k^X(x)) &= \partial^{\operatorname{Hom}'(Y,Z)_k(g_k(x))} \\ &= (\partial_{k+i}^Z \circ g_k(x)_i - (-1)^k g_k(x)_{i-1} \circ \partial_i^Y)_{i\in\mathbb{Z}} \\ g_{k-1}(\partial_k^X(x))_{n-k}(y) &= \partial_n^Z \circ g_k(x)_{n-k}(y) - (-1)^k g_k(x)_{n-k-1}(\partial_{n-k}^Y(y)).\end{aligned}$$

It follows:

$$\begin{aligned}\Psi(g)_{n-1} \circ \partial_n^{X\otimes Y}(x \otimes y) &= g_{k-1}(\partial_k^X(x))_{n-k}(y) + (-1)^k g_k(x)_{n-k-1}(\partial_{n-k}^Y(y)) \\ &= \partial_n^Z \circ g_k(x)_{n-k}(y) - (-1)^k g_k(x)_{n-k-1}(\partial_{n-k}^Y(y)) + (-1)^k g_k(x)_{n-k-1}(\partial_{n-k}^Y(y)) \\ &= \partial_n^Z \circ \Psi(g)_n(x \otimes y).\end{aligned}$$

Hence, we have a chain map $\Psi(g) : X \otimes Y \longrightarrow Z$. Now we construct an inverse

$$\Phi : \operatorname{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X \otimes Y, Z) \longrightarrow \operatorname{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X, \operatorname{Hom}'(Y, Z)).$$

Consider a chain map $f : X \otimes Y \longrightarrow Z$. For every $n \in \mathbb{Z}$, $\Phi(f)_n$ need to be a map $X_n \longrightarrow \prod_{k\in\mathbb{Z}} \operatorname{Hom}_R(Y_k, Z_{n+k})$. So for every $x \in X_n$, we define a family of maps $\Phi(f)_n(x) = (\Phi(f)_n(x)_k : Y_k \longrightarrow Z_{n+k})_{k\in\mathbb{Z}}$. Set

$$\boxed{\Phi(f)_n(x)_k(y) := f_{n+k}(x \otimes y) \text{ for every } x \otimes y \in X_n \otimes_R Y_k}$$

We need to show that the following square commutes:

$$\begin{array}{ccc} X_n & \xrightarrow{\quad \Phi(f)_n \quad} & \prod_{k\in\mathbb{Z}} \operatorname{Hom}_R(Y_k, Z_{n+k}) \\ {\scriptstyle \partial_n^X}\big\downarrow & & \big\downarrow{\scriptstyle \partial_n^{\operatorname{Hom}'(Y,Z)}} \\ X_{n-1} & \xrightarrow[\quad \Phi(f)_{n-1} \quad]{} & \operatorname{Hom}_R(Y_k, Z_{n+k-1}) \end{array}$$

We have:

$$\partial_n^{\operatorname{Hom}'(Y,Z)} \circ \Phi(f)_n(x) = (\partial_{n+k}^Z \circ \Phi(f)_n(x)_k - (-1)^n \Phi(f)_n(x)_{k-1} \circ \partial_k^Y)_{k\in\mathbb{Z}}.$$

For each $k \in \mathbb{Z}$ and each $y \in Y_k$, we get:

$$\begin{aligned}
\partial_n^{\mathrm{Hom}'(Y,Z)} \circ \Phi(f)_n(x)_k(y) &= \partial_{n+1}^Z \circ \Phi(f)_n(x)_k(y) - (-1)^n \Phi(f)_n(x)_{k-1}(\partial_k^Y(y)) \\
&= \partial_{n+k}^Z(f_{n+k}(x \otimes y)) - (-1)^n f_{n+k-1}(x \otimes \partial_k^Y(y)) \\
&= f_{n+k-1}(\partial_{n+k}^{X\otimes Y}(x \otimes y)) - (-1)^n f_{n+k-1}(x \otimes \partial_k^Y(y)) \\
&= f_{n+k-1}(\partial_n^X(x) \otimes y + (-1)^n x \otimes \partial_k^Y(y)) - (-1)^n f_{n+k-1}(x \otimes \partial_k^Y(y)) \\
&= f_{n+k-1}(\partial_n^X(x) \otimes y) + (-1)^n f_{n+k-1}(x \otimes \partial_k^Y(y)) - (-1)^n f_{n+k-1}(x \otimes \partial_k^Y(y)) \\
&= f_{n+k-1}(\partial_n^X(x) \otimes y) \\
&= \Phi(f)_{n-1}(\partial_n^X(x))_k(y)
\end{aligned}$$

Then, $\partial_n^{\mathrm{Hom}'(Y,Z)} \circ \Phi(f)_n(x) = \Phi(f)_{n-1}(\partial_n^X(x))$ for every $x \in X_n$. It follows

$$\partial_n^{\mathrm{Hom}'(Y,Z)} \circ \Phi(f)_n = \Phi(f)_{n-1} \circ \partial_n^X,$$

i.e. $\Phi(f)$ is a chain map.

Finally, we check that Φ and Ψ are the inverse of each other. We first show $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X\otimes Y,Z)}$. Consider a chain map $f : X \otimes Y \longrightarrow Z$. Then $\Phi(f)$ is a chain map $X \longrightarrow \mathrm{Hom}'(Y,Z)$, and so we have a family of maps $\Psi(\Phi(f))_n : (X \otimes Y)_n \longrightarrow Z_n$. For every $x \otimes y \in X_k \otimes_R Y_{n-k}$, we have

$$\Psi(\Phi(f))_n(x \otimes y) = \Phi(f)_k(x)_{n-k}(y) = f_n(x \otimes y).$$

It follows $(\Psi \circ \Phi)(f) = f$.

To check the equality $\Phi \circ \Psi = \mathrm{id}_{\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X,\mathrm{Hom}'(Y,Z))}$, consider a chain map $g : X \longrightarrow \mathrm{Hom}'(Y,Z)$. Then we have a family of maps $\Phi(\Psi(g))_n : X_n \longrightarrow \mathrm{Hom}'(Y,Z)_n$, and so

$$\Phi(\Psi(g))_n(x)_k(y) = \Psi(g)_{n+k}(x \otimes y) = g_n(x)_k(y),$$

for every $x \in X_n$ and $y \in Y_k$. It follows $(\Phi \circ \Psi)(g) = g$.

Now we focus on showing that the isomorphism Φ is natural. We need to recall first the functoriality of $- \otimes -$ and $\mathrm{Hom}'(-,-)$.

(i) Definition of $f^{\mathrm{op}} \otimes g^{\mathrm{op}}$: We consider the tensor product of chain complexes as a functor of the form $\mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}} \times \mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}} \longrightarrow \mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}}$. Consider two chain maps $f^{\mathrm{op}} : X^{\mathrm{op}} \longrightarrow (X')^{\mathrm{op}}$ and $g^{\mathrm{op}} : Y^{\mathrm{op}} \longrightarrow (Y')^{\mathrm{op}}$. The chain map $f^{\mathrm{op}} \otimes g^{\mathrm{op}} : X^{\mathrm{op}} \otimes Y^{\mathrm{op}} \longrightarrow (X')^{\mathrm{op}} \otimes (Y')^{\mathrm{op}}$ is defined for each $n \in \mathbb{Z}$ as the map

$$\begin{aligned}
(f \otimes g)_n : \bigoplus_{k\in\mathbb{Z}} X'_k \otimes_R Y'_{n-k} &\longrightarrow \bigoplus_{k\in\mathbb{Z}} X_k \otimes_R Y_{n-k} \\
x' \otimes y' &\mapsto f_k(x') \otimes g_{n-k}(y') \text{ for every } x' \otimes y' \in X'_k \otimes_R Y'_{n-k}.
\end{aligned}$$

We verify that $f \otimes g$ is indeed a chain map, i.e. that the following diagram commutes:

$$\begin{array}{ccc}
\bigoplus_{k\in\mathbb{Z}} X'_k \otimes_R Y'_{n-k} & \xrightarrow{(f\otimes g)_n} & \bigoplus_{k\in\mathbb{Z}} X_k \otimes_R Y_{n-k} \\
\Big\downarrow {\scriptstyle \partial_n^{X'\otimes Y'}} & & \Big\downarrow {\scriptstyle \partial_n^{X\otimes Y}} \\
\bigoplus_{k\in\mathbb{Z}} X'_k \otimes_R Y'_{n-1-k} & \xrightarrow[(f\otimes g)_{n-1}]{} & \bigoplus_{k\in\mathbb{Z}} X_k \otimes_R Y_{n-1-k}
\end{array}$$

Let $x' \otimes y' \in X'_k \otimes_R Y'_{n-k}$. We have:

$$\begin{aligned}
\partial_n^{X\otimes Y} \circ (f \otimes g)_n(x' \otimes y') &= \partial_n^{X\otimes Y}(f_k(x') \otimes g_{n-k}(y')) \\
&= \partial_k^X \circ f_k(x') \otimes g_{n-k}(y') + (-1)^k f_k(x') \otimes \partial_{n-k}^Y \circ g_{n-k}(y') \\
&= f_{k-1} \circ \partial_k^{X'}(x') \otimes g_{n-k}(y') + (-1)^k f_k(x') \otimes g_{n-1-k} \circ \partial_{n-k}^{Y'}(y') \\
&= (f \otimes g)_{n-1}(\partial_k^{X'}(x') \otimes y') + (-1)^k (f \otimes g)_{n-1}(x \otimes \partial_{n-k}^{Y'}(y')) \\
&= (f \otimes g)_{n-1}(\partial_k^{X'}(x') \otimes y' + (-1)^k x \otimes \partial_{n-k}^{Y'}(y')) \\
&= (f \otimes g)_{n-1} \circ \partial_n^{X'\otimes Y'}(x' \otimes y').
\end{aligned}$$

(ii) <u>Definition of $\mathrm{Hom}'(g^{\mathrm{op}}, h)$</u>: Consider the above chain map $g^{\mathrm{op}} : Y^{\mathrm{op}} \longrightarrow (Y')^{\mathrm{op}}$ along with another chain map $h : Z \longrightarrow Z'$. We construct a chain map $\mathrm{Hom}'(g^{\mathrm{op}}, h) : \mathrm{Hom}'(Y^{\mathrm{op}}, Z) \longrightarrow \mathrm{Hom}'((Y')^{\mathrm{op}}, Z')$. For each $n \in \mathbb{Z}$, the homomorphism $\mathrm{Hom}'(g^{\mathrm{op}}, h)_n : \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(Y_k, Z_{n+k}) \longrightarrow \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(Y'_k, Z'_{n+k})$ is defined as follows: for every family $(\alpha_k : Y_k \longrightarrow Z_{n+k})_{k\in\mathbb{Z}}$, $\mathrm{Hom}'(g^{\mathrm{op}}, h)_n((\alpha_k)_{k\in\mathbb{Z}})$ is the family given by $(h_{n+k} \circ \alpha_k \circ g_k : Y'_k \longrightarrow Z'_{n+k})_{k\in\mathbb{Z}}$.

$$\begin{array}{ccc}
Y_k & \xrightarrow{\quad \alpha_k \quad} & Z_{n+k} \\
{\scriptstyle g_k}\uparrow & & \downarrow{\scriptstyle h_{n_k}} \\
Y'_k & \xrightarrow[\mathrm{Hom}'(g^{\mathrm{op}}, h)_n((\alpha_k)_{k\in\mathbb{Z}})_k]{} & Z'_{n+k}
\end{array}$$

We show that the following diagram commutes:

$$\begin{array}{ccc}
\prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(Y_k, Z_{n+k}) & \xrightarrow{\mathrm{Hom}'(g^{\mathrm{op}}, h)_n} & \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(Y'_k, Z'_{n+k}) \\
{\scriptstyle \partial_n^{\mathrm{Hom}'(Y^{\mathrm{op}}, Z)}}\downarrow & & \downarrow{\scriptstyle \partial_n^{\mathrm{Hom}'((Y')^{\mathrm{op}}, Z')}} \\
\prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(Y_k, Z_{n-1+k}) & \xrightarrow[\mathrm{Hom}'(g^{\mathrm{op}}, h)_{n-1}]{} & \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(Y'_k, Z'_{n+k-1})
\end{array}$$

Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}} \in \prod_{k\in\mathbb{Z}} \mathrm{Hom}_R(Y_k, Z_{n+k})$. We have:

$$\begin{aligned}
\partial_n^{\mathrm{Hom}'((Y')^{\mathrm{op}}, Z')} \circ \mathrm{Hom}'(g^{\mathrm{op}}, h)_n(\alpha) &= \partial_n^{\mathrm{Hom}'((Y')^{\mathrm{op}}, Z')}((h_{k+n} \circ \alpha_k \circ g_k)_{k\in\mathbb{Z}}) \\
&= (\partial_{n+k}^{Z'} \circ h_{n+k} \circ \alpha_k \circ g_k - (-1)^n h_{n+k-1} \circ \alpha_{k-1} \circ g_{k-1} \circ \partial_k^{Y'})_{k\in\mathbb{Z}} \\
&= (h_{n+k-1} \circ \partial_{n+k}^{Z} \circ \alpha_k \circ g_k - (-1)^n h_{n+k-1} \circ \alpha_{k-1} \circ \partial_k^{Y} \circ g_k)_{k\in\mathbb{Z}} \\
&= (h_{n+k-1} \circ (\partial_{n+k}^{Z} \circ \alpha_k - (-1)^n \alpha_{k-1} \circ \partial_k^{Y}) \circ g_k)_{k\in\mathbb{Z}} \\
&= (h_{n+k-1} \circ (\partial_n^{\mathrm{Hom}'(Y^{\mathrm{op}}, Z)}(\alpha))_k \circ g_k)_{k\in\mathbb{Z}} \\
&= \mathrm{Hom}'(g^{\mathrm{op}}, h)_{n-1} \circ \partial_n^{\mathrm{Hom}'(Y^{\mathrm{op}}, Z)}(\alpha).
\end{aligned}$$

So far we have two bifunctors

$$\begin{gathered}
- \otimes - : \mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}} \times \mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}} \longrightarrow \mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}}, \\
\mathrm{Hom}'(-, -) : \mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}} \times \mathbf{Ch}({}_R\mathbf{Mod}) \longrightarrow \mathbf{Ch}({}_R\mathbf{Mod}).
\end{gathered}$$

(iii) Definition of $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \mathrm{Hom}'(g^{\mathrm{op}}, h))$: Note that $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \mathrm{Hom}'(g^{\mathrm{op}}, h))$ is a map $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(X^{\mathrm{op}}, \mathrm{Hom}'(Y^{\mathrm{op}}, Z)) \longrightarrow \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}((X')^{\mathrm{op}}, \mathrm{Hom}'((Y')^{\mathrm{op}}, Z'))$. Given a chain map $\beta : X^{\mathrm{op}} \longrightarrow \mathrm{Hom}'(Y^{\mathrm{op}}, Z)$, $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \mathrm{Hom}'(g^{\mathrm{op}}, h))(\beta) := \mathrm{Hom}'(g^{\mathrm{op}}, h) \circ \beta \circ f^{\mathrm{op}}$.

$$\begin{array}{ccc}
X & \xrightarrow{\quad\beta\quad} & \mathrm{Hom}'(Y^{\mathrm{op}}, Z) \\
{\scriptstyle f}\uparrow & & \downarrow{\scriptstyle \mathrm{Hom}'(g^{\mathrm{op}}, h)} \\
X' & \xrightarrow[\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \mathrm{Hom}'(g^{\mathrm{op}}, h))(\beta)]{} & \mathrm{Hom}'((Y^{\mathrm{op}})', Z')
\end{array}$$

(iv) Definition of $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}} \otimes g^{\mathrm{op}}, h))$: Note that $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}} \otimes g^{\mathrm{op}}, h))$ is a map of the form $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(X^{\mathrm{op}} \otimes Y^{\mathrm{op}}, Z) \longrightarrow \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}((X')^{\mathrm{op}} \otimes (Y')^{\mathrm{op}}, Z')$. Given a chain map $\gamma : X^{\mathrm{op}} \otimes Y^{\mathrm{op}} \longrightarrow Z$, $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}} \otimes g^{\mathrm{op}}, h))(\gamma) := h \circ \gamma \circ (f \otimes g)$.

$$\begin{array}{ccc}
X \otimes Y & \xrightarrow{\quad\gamma\quad} & Z \\
{\scriptstyle f \otimes g}\uparrow & & \downarrow{\scriptstyle h} \\
X' \otimes Y' & \xrightarrow[\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}} \otimes g^{\mathrm{op}}, h))(\gamma)]{} & Z'
\end{array}$$

Now we show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(X^{\mathrm{op}}, \mathrm{Hom}'(Y^{\mathrm{op}}, Z)) & \xrightarrow{\Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}} & \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(X^{\mathrm{op}} \otimes Y^{\mathrm{op}}, Z) \\
{\scriptstyle \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \mathrm{Hom}'(g^{\mathrm{op}}, h))}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}} \otimes g^{\mathrm{op}}, h)} \\
\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}((X')^{\mathrm{op}}, \mathrm{Hom}'((Y')^{\mathrm{op}}, Z')) & \xrightarrow[\Psi_{(X')^{\mathrm{op}}, (Y')^{\mathrm{op}}, Z'}]{} & \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}((X')^{\mathrm{op}} \otimes (Y')^{\mathrm{op}}, Z)
\end{array}$$

For every chain map $\beta : X \longrightarrow \mathrm{Hom}'(Y^{\mathrm{op}}, Z)$, we compute $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}} \otimes g^{\mathrm{op}}, h) \circ \Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}(\beta)$ and $\Psi_{(X')^{\mathrm{op}}, (Y')^{\mathrm{op}}, Z'} \circ \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \mathrm{Hom}'(g^{\mathrm{op}}, h))(\beta)$. Let $x' \otimes y' \in X'_k \otimes_R Y'_{n-k}$, we have:

$$\begin{aligned}
[\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}} \otimes g^{\mathrm{op}}, h) \circ \Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}(\beta)]_n(x' \otimes y') &= [h \circ \Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}(\beta) \circ (f \otimes g)]_n(x' \otimes y') \\
&= [h_n \circ \Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}(\beta)_n \circ (f \otimes g)_n](x' \otimes y') \\
&= h_n \circ \Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}(\beta)_n(f_k(x') \otimes g_{n-k}(y')) \\
&= h_n(\beta_k(f_k(x')))_{n-k}(g_{n-k}(y')), \\
[\Psi_{(X')^{\mathrm{op}}, (Y')^{\mathrm{op}}, Z'} \circ \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \mathrm{Hom}'(g^{\mathrm{op}}, h))(\beta)]_n(x' \otimes y') &= \Psi_{(X')^{\mathrm{op}}, (Y')^{\mathrm{op}}, Z'}(\mathrm{Hom}'(g^{\mathrm{op}}, h) \circ \beta \circ f)_n(x' \otimes y') \\
&= (\mathrm{Hom}'(g^{\mathrm{op}}, h) \circ \beta \circ f)_k(x')_{n-k}(y') \\
&= \mathrm{Hom}'(g^{\mathrm{op}}, h)_k(\beta_k(f_k(x')))_{n-k}(y') \\
&= h_n \circ \beta_k(f_k(x'))_{n-k} \circ g_{n-k}(y') \\
&= h_n(\beta_k(f_k(x')))_{n-k}(g_{n-k}(y')).
\end{aligned}$$

Similarly, one can show that Φ is also natural.

3 The bar tensor product of chain complexes

Given to chain complexes $X, Y \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$, the **bar tensor product** of X and Y is the chain complex $X\overline{\otimes}Y$ defined by

$$\boxed{(X\overline{\otimes}Y)_n := \frac{(X\otimes Y)_n}{B_n(X\otimes Y)}}$$

where the boundary maps $\partial_n^{X\overline{\otimes}Y} : (X\overline{\otimes}Y)_n \longrightarrow (X\overline{\otimes}Y)_{n-1}$ are given by

$$\boxed{x \otimes y + B_n(X \otimes Y) \mapsto \partial^X(x) \otimes y + B_{n-1}(X \otimes Y)}$$

It turns out that $(\mathbf{Ch}({}_R\mathbf{Mod}), \overline{\otimes})$ is a symmetric monoidal category, where the unit is given by the 1-disk complex $D^1(R)$ defined by

$$\boxed{(D^1(R))_k := \begin{cases} R & \text{if } k = 0 \text{ or } 1, \\ 0 & \text{otherwise.} \end{cases}}$$

where the boundary maps are all zero except for $(\partial^{D^1(R)})_1 = \mathrm{id}_R$ (See [1, Proposition 4.2.1 4] for details).
The bar-Hom complex of X and Y is defined as the chain complex $\overline{\mathrm{Hom}}(X, Y)$ given by

$$\boxed{\overline{\mathrm{Hom}}(X, Y)_n = Z_n(\mathrm{Hom}'(X, Y))}$$

whose boundary maps $\partial_n^{\overline{\mathrm{Hom}}(X,Y)} : \overline{\mathrm{Hom}}(X, Y)_n \longrightarrow \overline{\mathrm{Hom}}(X, Y)_{n-1}$ are given by

$$\boxed{f = (f_k : X_k \longrightarrow Y_{n+k})_{k\in\mathbb{Z}} \mapsto (\partial_{n+k}^Y \circ f_k : X_k \longrightarrow Y_{n+k-1})_{k\in\mathbb{Z}}}$$

The goal of this section is to show that the bar-him functor $\overline{\mathrm{Hom}}_{\mathbf{Ch}({}_R\mathbf{Mod})}(-,-)$ is the internal Hom of $(\mathbf{Ch}({}_R\mathbf{Mod}), \overline{\otimes})$.

Theorem 3.1

Let R be a commutative ring with unit, and $X, Y, Z \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$. Then there exists a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X, \overline{\mathrm{Hom}}(Y, Z)) \cong \mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X\overline{\otimes}Y, Z)$$

We construct an isomorphism

$$\Phi : \mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X\overline{\otimes}Y, Z) \longrightarrow \mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X, \overline{\mathrm{Hom}}(Y, Z)).$$

Consider a chain map $f : X\overline{\otimes}Y \longrightarrow Z$. We need a chain map $\Phi(f) : X \longrightarrow \overline{\mathrm{Hom}}(Y, Z)$. Fix $n \in \mathbb{Z}$ and let $x \in X_n$. We want to construct a family of maps $\Phi(f)_n(x) = (\Phi(f)_n(x)_k : Y_k \longrightarrow Z_{n+k})_{k\in\mathbb{Z}}$ such that $\partial_n^{\mathrm{Hom}'(Y,Z)}(\Phi(f)_n) = 0$, i.e.

$$\partial_{n+k}^Z \circ \Phi(f)_n(x)_k = (-1)^n \Phi(f)_n(x)_{k-1} \circ \partial_k^Y,$$

for every $k \in \mathbb{Z}$. We have a homomorphisms $f_{n+k} : \frac{(X\otimes Y)_{n+k}}{B_{n+k}(X\otimes Y)} \longrightarrow Z_{n+k}$. Define

$$\boxed{\Phi(f)_n(x)_k(y) := (-1)^k f_{n+k}(x \otimes y + B_n(X \otimes Y))}$$

Since f is a chain map, the following square commutes:

$$\begin{array}{ccc}
\frac{(X\otimes Y)_{n+k}}{B_{n+k}(X\otimes Y)} & \xrightarrow{f_{n+k}} & Z_{n+k} \\
\partial^{X\overline{\otimes}Y}_{n+k} \Big\downarrow & & \Big\downarrow \partial^{Z}_{n+k} \\
\frac{(X\otimes Y)_{n+k}}{B_{n+k}(X\otimes Y)} & \xrightarrow[f_{n+k-1}]{} & Z_{n+k-1}
\end{array}$$

Then we have:

$$\begin{aligned}
\partial^Z_{n+k} \circ \Phi(f)_n(x)_k(y) &= (-1)^k \partial^Z_{n+k} \circ f_{n+k}(x \otimes y + B_n(X \otimes Y)) \\
&= (-1)^k f_{n+k-1} \circ \partial^{X\overline{\otimes}Y}_{n+k}(x \otimes y + B_{n+k}(X \otimes Y)) \\
&= (-1)^k f_{n+k-1}(\partial^X_n(x) \otimes y + B_{n+k-1}(X \otimes Y))
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{n-1}(-1)^k f_{n+k-1}(x \otimes \partial^Y_k(y) + B_{n+k-1}(X \otimes Y)), \\
&\quad \text{since } \partial^{X\otimes Y}_{n+k}(x \otimes y) = \partial^X_n(x) \otimes y + (-1)^n x \otimes \partial^Y_k(y), \\
&= (-1)^n(-1)^{k-1} f_{n+k-1}(x \otimes \partial^Y_k(y) + B_{n+k-1}(X \otimes Y)) \\
&= (-1)^n \Phi(f)_n(x)_{k-1}(\partial^Y_k(y)).
\end{aligned}$$

It follows $\Phi(f)_n \in Z_n(\mathrm{Hom}'(Y, Z))$. Now we check $\Phi(f)$ is a chain map, i.e. that the following square commutes for each $n \in \mathbb{Z}$:

$$\begin{array}{ccc}
X_n & \xrightarrow{\Phi(f)_n} & Z_n(\mathrm{Hom}'(Y,Z)) \\
\partial^X_n \Big\downarrow & & \Big\downarrow \partial^{\overline{\mathrm{Hom}}(Y,Z)}_n \\
X_{n-1} & \xrightarrow[\Phi(f)_{n-1}]{} & Z_{n-1}(\mathrm{Hom}'(Y,Z))
\end{array}$$

Let $x \in X_n$. Then

$$\begin{aligned}
\partial^{\overline{\mathrm{Hom}}(Y,Z)}_n \circ \Phi(f)_n(x) &= \partial^{\overline{\mathrm{Hom}}(Y,Z)}_n((\Phi(f)_n(x)_k)_{k\in\mathbb{Z}}) = (\partial^Z_{n+k} \circ \Phi(f)_n(x)_k)_{k\in\mathbb{Z}}, \\
\Phi(f)_{n-1}(\partial^X_n(x)) &= (\Phi(f)_{n-1}(\partial^X_n(x))_k)_{k\in\mathbb{Z}}.
\end{aligned}$$

Now let $y \in Y_k$. We have:

$$\begin{aligned}
\partial^Z_{n+k} \circ \Phi(f)_n(x)_k(y) &= (-1)^k \partial^Z_{n+k}(f_{n+k}(x \otimes y + B_{n+k}(X \otimes Y))) \\
&= (-1)^k f_{n+k-1}(\partial^X_n(x) \otimes y + B_{n+k-1}(X \otimes Y)) \\
&= \Phi(f)_{n-1}(\partial^X_n(x))_k(y).
\end{aligned}$$

It follows $\partial^{\overline{\mathrm{Hom}}(Y,Z)}_n \circ \Phi(f)_n = \Phi(f)_{n-1} \circ \partial^X_n$.

Now we construct an inverse

$$\Psi : \mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X, \overline{\mathrm{Hom}}(Y, Z)) \longrightarrow \mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X\overline{\otimes}Y, Z).$$

Consider a chain map $g : X \longrightarrow \overline{\mathrm{Hom}}(Y, Z)$. We want a chain map $\Psi(g) : X\overline{\otimes}Y \longrightarrow Z$. For every $n \in \mathbb{Z}$, we define homomorphisms $\Psi(g)_n : \frac{(X\otimes Y)_n}{B_n(X\otimes Y)} \longrightarrow Z_n$. Let $x \otimes y + B_n(X \otimes Y) \in (X\overline{\otimes}Y)_n$, with $x \in X_n$ and $y \in Y_{n-k}$. Set

$$\boxed{\Psi(g)_n(x \otimes y + B_n(X \otimes Y)) := (-1)^{n-k} g_k(x)_{n-k}(y)}$$

First, we need to check that $\Psi(g)_n$ is well defined. We only study the case

$$x \otimes y - x' \otimes y' = \partial_{n+1}^{X\otimes Y}(c), \text{ for some } c \in (X \otimes Y)_{n+1}.$$

Write $c = \sum_{i+j=n+1} x^i \otimes y^j$, where $x^i \in X_i$ and $y^j \in Y_j$. We have

$$x \otimes y - x' \otimes y' = \sum_{i+j=n+1} \partial_{n+1}^{X\otimes Y}(x^i \otimes y^j) = \sum_{i+j=n+1} (\partial_i^X(x^i) \otimes y^j + (-1)^i x^i \otimes \partial_j^Y(y)),$$

$$(-1)^{n-k} g_k(x)_{n-k}(y) - (-1)^{n-k} g_k(x')_{n-k}(y') = \sum_{i+j=n+1} (-1)^j g_{i-1}(\partial_i^X(x^i))_j(y^j) + (-1)^i(-1)^{j-1} g_i(x^i)_{j-1}(\partial_j^Y(y^j)).$$

Since g is a chain map, the following square commutes:

$$\begin{array}{ccc} X_i & \xrightarrow{\ g_i\ } & Z_i(\mathrm{Hom}'(Y,Z)) \\ {\scriptstyle \partial_i^X}\downarrow & & \downarrow{\scriptstyle \partial_i^{\overline{\mathrm{Hom}}(Y,Z)}} \\ X_{i-1} & \xrightarrow[\ g_{i-1}\ ]{} & Z_{i-1}(\mathrm{Hom}'(Y,Z)) \end{array}$$

Then:

$$\begin{aligned} (-1)^{n-k} g_k(x)_{n-k}(y) - (-1)^{n-k} g_k(x')_{n-k}(y') &= \sum_{i+j=n+1} (-1)^j g_{i-1}(\partial_i^X(x^i))_j(y^j) + (-1)^i(-1)^{j-1} g_i(x^i)_{j-1}(\partial_j^Y(y^j)) \\ &= \sum_{i+j=n+1} (-1)^j (\partial_i^{\overline{\mathrm{Hom}}(Y,Z)}(g_i(x^i)))_j(y^j) + (-1)^{i+j-1} g_i(x^i)_{j-1}(\partial_j^Y(y^j)) \\ &= \sum_{i+j=n+1} (-1)^j \partial_{j+i}^Z \circ g_i(x^i)_j(y^j) + (-1)^{i+j-1} g_i(x^i)_{j-1}(\partial_j^Y(y^j)). \end{aligned}$$

On the other hand, $g_i(x^i) \in Z_i(\mathrm{Hom}'(Y,Z))$, then for every j we have $\partial_{j+i}^Z \circ g_i(x^i)_j = (-1)^i g_i(x^i)_{j-1} \circ \partial_j^Y$.

$$\begin{aligned} (-1)^{n-k} g_k(x)_{n-k}(y) - (-1)^{n-k} g_k(x')_{n-k}(y') &= \sum_{i+j=n+1} (-1)^j \partial_{j+i}^Z \circ g_i(x^i)_j(y^j) + (-1)^{i+j-1} g_i(x^i)_{j-1}(\partial_j^Y(y^j)) \\ &= \sum_{i+j=n+1} (-1)^{j+i} g_i(x^i)_{j-1}(\partial_j^Y(y^j)) + (-1)^{i+j-1} g_i(x^i)_{j-1}(\partial_j^Y(y^j)) \\ &= 0. \end{aligned}$$

Hence,

$$\Psi(g)_n(x \otimes y + B_n(X \otimes Y)) = \Psi(g)_n(x' \otimes y' + B_n(X \otimes Y)).$$

Now we check that $\Psi(g)$ defines a chain map $X\overline{\otimes}Y \longrightarrow Z$. For every $n \in \mathbb{Z}$, we show that the following square commutes:

$$\begin{array}{ccc}
\frac{(X\otimes Y)_n}{B_n(X\otimes Y)} & \xrightarrow{\Psi(g)_n} & Z_n \\
\Big\downarrow{\scriptstyle \partial_n^{X\overline{\otimes}Y}} & & \Big\downarrow{\scriptstyle \partial_n^{Z}} \\
\frac{(X\otimes Y)_{n-1}}{B_{n-1}(X\otimes Y)} & \xrightarrow[\Psi(g)_{n-1}]{} & Z_{n-1}
\end{array}$$

Let $x \otimes y + B_n(X \otimes Y) \in \frac{(X\otimes Y)_n}{B_n(X\otimes Y)}$. We have:

$$\begin{aligned}
\partial_n^Z \circ \Psi(g)_n(x \otimes y + B_n(X \otimes Y)) &= \partial_n^Z((-1)^{n-k} g_k(x)_{n-k}(y)) \\
&= (-1)^{n-k} \partial_n^Z(g_k(x)_{n-k})(y) \\
&= (-1)^{n-k} g_{k-1}(\partial_k^X(x))_{n-k}(y), \text{ since } g \text{ is a chain map} \\
&= \Psi(g)_{n-1}(\partial_k^X(x) \otimes y) \\
&= \Psi(g)_{n-1} \circ \partial_n^{X\overline{\otimes}Y}(x \otimes y + B_n(X \otimes Y)).
\end{aligned}$$

It is only left to show that $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X\overline{\otimes}Y,Z)}$ and $\Phi \circ \Psi = \mathrm{id}_{\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(X,\overline{\mathrm{Hom}}(Y,Z))}$. Consider two chain maps $f : X\overline{\otimes}Y \longrightarrow Z$ and $g : X \longrightarrow \overline{\mathrm{Hom}}(Y, Z)$. We have:

$$\begin{aligned}
(\Psi \circ \Phi)(f)_n(x \otimes y + B_n(X \otimes Y)) &= (-1)^{n-k} \Phi(f)_k(x)_{n-k}(y) \\
&= (-1)^{n-k}(-1)^{n-k} f_n(x \otimes y + B_n(X \otimes Y)) \\
&= f_n(x \otimes y + B_n(X \otimes Y)), \\
(\Phi \circ \Psi)(g)_n(x)_k(y) &= (-1)^k \Psi(g)_{n+k}(x \otimes y + B_n(X \otimes Y)) \\
&= (-1)^k(-1)^k g_n(x)_k(y) \\
&= g_n(x)_k(y).
\end{aligned}$$

Therefore, Theorem 3.1 follows.

Before proving that Ψ is natural, we recall some functorial constructions for $-\overline{\otimes}-$ and $\overline{\mathrm{Hom}}(-,-)$.

(i) Definition of $f\overline{\otimes}g$: We consider two chain maps $f^{\mathrm{op}} : X^{\mathrm{op}} \longrightarrow (X')^{\mathrm{op}}$ and $g^{\mathrm{op}} : Y^{\mathrm{op}} \longrightarrow (Y')^{\mathrm{op}}$. The chain map $f^{\mathrm{op}}\overline{\otimes}g^{\mathrm{op}} : X^{\mathrm{op}}\overline{\otimes}Y^{\mathrm{op}} \longrightarrow (X')^{\mathrm{op}}\overline{\otimes}(Y')^{\mathrm{op}}$ is given at each $n \in \mathbb{Z}$ by:

$$\begin{aligned}
(f\overline{\otimes}g)_n : \frac{(X' \otimes Y')_n}{B_n(X' \otimes Y')} &\longrightarrow \frac{(X \otimes Y)_n}{B_n(X \otimes Y)}, \\
x' \otimes y' + B_n(X' \otimes Y') &\mapsto f_k(x') \otimes g_{n-k}(y') + B_n(X \otimes Y)
\end{aligned}$$

This map is well defined, since it turns out to be the map induced by the universal property of cokernels in the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_n(X' \otimes Y') & \longrightarrow & (X' \otimes Y)_n & \longrightarrow & (X'\overline{\otimes}Y')_n & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle (f\otimes g)_n|_{B_n(X'\otimes Y')}} & & \Big\downarrow{\scriptstyle (f\otimes g)_n} & & \vdots{\scriptstyle (f\overline{\otimes}g)_n} & & \\
0 & \longrightarrow & B_n(X \otimes Y) & \longrightarrow & (X \otimes Y)_n & \longrightarrow & (X\overline{\otimes}Y)_n & \longrightarrow & 0
\end{array}$$

Now we check that the following diagram commutes:

$$\begin{array}{ccc} \frac{(X'\otimes Y')_n}{B_n(X'\otimes Y')} & \xrightarrow{(f\overline{\otimes} g)_n} & \frac{(X\otimes Y)_n}{B_n(X\otimes Y)} \\ \partial_n^{X'\overline{\otimes}Y'} \Big\downarrow & & \Big\downarrow \partial_n^{X\overline{\otimes}Y} \\ \frac{(X'\otimes Y')_{n-1}}{B_{n-1}(X'\otimes Y')} & \xrightarrow[(f\overline{\otimes}g)_{n-1}]{} & \frac{(X\otimes Y)_{n-1}}{B_{n-1}(X\otimes Y)} \end{array}$$

Let $x'\otimes y' + B_n(X'\otimes Y') \in \frac{(X'\otimes Y')_n}{B_n(X'\otimes Y')}$. We have:

$$\begin{aligned} \partial_n^{X\overline{\otimes}Y} \circ (f\overline{\otimes}g)_n(x'\otimes y' + B_n(X\otimes Y)) &= \partial_n^{X\overline{\otimes}Y}(f_k(x')\otimes g_{n-k}(y') + B_n(X'\otimes Y')) \\ &= \partial_k^X(f_k(x'))\otimes g_{n-k}(y') + B_{n-1}(X\otimes Y) \\ &= f_{k-1}(\partial_k^{X'}(x'))\otimes g_{n-k}(y') + B_{n-1}(X\otimes Y) \\ &= (f\otimes g)_{n-1}(\partial_k^{X'}(x')\otimes y') + B_{n-1}(X\otimes Y) \\ &= (f\overline{\otimes}g)_{n-1}(\partial_k^{X'}(x')\otimes y' + B_{n-1}(X'\otimes Y')) \\ &= (f\overline{\otimes}g)_{n-1}\circ \partial_n^{X'\overline{\otimes}Y'}(x'\otimes y' + B_n(X'\otimes Y')) \end{aligned}$$

(ii) Definition of $\overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)$: Consider two chain maps $g^{\mathrm{op}} : Y^{\mathrm{op}} \longrightarrow (Y')^{\mathrm{op}}$ and $h : Z \longrightarrow Z'$. We construct a chain map $\overline{\mathrm{Hom}}(g^{\mathrm{op}}, h) : \overline{\mathrm{Hom}}(Y^{\mathrm{op}}, Z) \longrightarrow \overline{\mathrm{Hom}}((Y')^{\mathrm{op}}, Z')$. For each $n\in\mathbb{Z}$, we define a homomorphism $\overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)_n : Z_n(\mathrm{Hom}'(Y^{\mathrm{op}}, Z)) \longrightarrow Z_n(\mathrm{Hom}'((Y')^{\mathrm{op}}, Z'))$. Let $\alpha = (\alpha_k)_{k\in\mathbb{Z}} \in Z_n(\mathrm{Hom}'(Y^{\mathrm{op}}, Z))$, then $\partial^Z_{n+k}\circ\alpha_k = (-1)^n\alpha_{k-1}\circ\partial^Y_k$ for every $k\in\mathbb{Z}$. We have

$$\begin{aligned} \partial^{Z'}_{n+k}\circ(h_{n+k}\circ\alpha_k\circ g_k) &= h_{n+k-1}\circ\partial^Z_{n+k}\circ\alpha_k\circ g_k = h_{n+k-1}\circ(-1)^n\alpha_{k-1}\circ\partial^Y_k\circ g_k \\ &= (-1)^n h_{n+k-1}\circ\alpha_{k-1}\circ g_{k-1}\circ\partial^{Y'}_k. \end{aligned}$$

Then $(h_{n+k}\circ\alpha_k\circ g_k)_{k\in\mathbb{Z}} \in Z_n(\mathrm{Hom}'((Y')^{\mathrm{op}}, Z'))$ and so it makes sense to set $\overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)_n(\alpha) := (h_{n+k}\circ\alpha_k\circ g_k)_{k\in\mathbb{Z}}$. Now we check that the following diagram commutes:

$$\begin{array}{ccc} Z_n(\mathrm{Hom}'(Y^{\mathrm{op}}, Z)) & \xrightarrow{\overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)_n} & Z_n(\mathrm{Hom}'((Y')^{\mathrm{op}}, Z')) \\ \partial_n^{\overline{\mathrm{Hom}}(Y^{\mathrm{op}}, Z)} \Big\downarrow & & \Big\downarrow \partial_n^{\overline{\mathrm{Hom}}((Y')^{\mathrm{op}}, Z')} \\ Z_{n-1}(\mathrm{Hom}'(Y^{\mathrm{op}}, Z)) & \xrightarrow[\overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)_{n-1}]{} & Z_{n-1}(\mathrm{Hom}'((Y')^{\mathrm{op}}, Z')) \end{array}$$

$$\begin{aligned} \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)_{n-1}\circ\partial_n^{\overline{\mathrm{Hom}}(Y^{\mathrm{op}}, Z)}(\alpha) &= \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)_{n-1}((\partial^Z_{n+k}\circ\alpha_k)_{k\in\mathbb{Z}}) \\ &= (h_{n+k-1}\circ\partial^Z_{n+k}\circ\alpha_k\circ g_k)_{k\in\mathbb{Z}} \\ &= (\partial^{Z'}_{n+k}\circ h_{n+k}\circ\alpha_k\circ g_k)_{k\in\mathbb{Z}} \\ &= \partial_n^{\overline{\mathrm{Hom}}((Y')^{\mathrm{op}}, Z')}\circ\overline{\mathrm{Hom}}_n(g^{\mathrm{op}}, h)_n(\alpha). \end{aligned}$$

(iii) Definition of $\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(f^{\mathrm{op}}, \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h))$: We have a functor $\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(-, \overline{\mathrm{Hom}}(-,-)) : \mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}}\times\mathbf{Ch}({}_R\mathbf{Mod})^{\mathrm{op}}\times\mathbf{Ch}({}_R\mathbf{Mod}) \longrightarrow \mathbf{Ab}$. Consider three chain maps $f^{\mathrm{op}} : X^{\mathrm{op}} \longrightarrow (X')^{\mathrm{op}}$, $g^{\mathrm{op}} : Y^{\mathrm{op}} \longrightarrow (Y')^{\mathrm{op}}$, and $h : Z \longrightarrow Z'$. The group homomorphism $\mathrm{Hom}_{\mathbf{Ch}({}_R\mathbf{Mod})}(f^{\mathrm{op}}, \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)) :$

$\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(X^{\mathrm{op}}, \overline{\mathrm{Hom}}(Y^{\mathrm{op}}, Z)) \longrightarrow \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}((X')^{\mathrm{op}}, \overline{\mathrm{Hom}}((Y')^{\mathrm{op}}, Z'))$ is given by

$$\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h))(\beta) := \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h) \circ \beta \circ f$$

for every $\beta : X^{\mathrm{op}} \longrightarrow \overline{\mathrm{Hom}}(Y^{\mathrm{op}}, Z)$.

$$\begin{array}{ccc} X & \xrightarrow{\quad\beta\quad} & \overline{\mathrm{Hom}}(Y^{\mathrm{op}}, Z) \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)} \\ X' & \xrightarrow[\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h))(\beta)]{} & \overline{\mathrm{Hom}}((Y')^{\mathrm{op}}, Z') \end{array}$$

(iv) Definition of $\mathrm{Hom}(f^{\mathrm{op}}\overline{\otimes}g^{\mathrm{op}}, h)$: $\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}\overline{\otimes}g^{\mathrm{op}}, h)(\gamma) := h \circ \gamma \circ (f\overline{\otimes}g)$, with f^{op}, g^{op} and h as above, and $\gamma : X\overline{\otimes}Y \longrightarrow Z$.

To show Ψ is natural, we verify the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(X^{\mathrm{op}}, \overline{\mathrm{Hom}}(Y^{\mathrm{op}}, Z)) & \xrightarrow{\Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}} & \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(X^{\mathrm{op}}\overline{\otimes}Y^{\mathrm{op}}, Z) \\ {\scriptstyle \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h))}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}\overline{\otimes}g^{\mathrm{op}}, h)} \\ \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}((X')^{\mathrm{op}}, \overline{\mathrm{Hom}}((Y')^{\mathrm{op}}, Z)) & \xrightarrow[\Psi_{(X')^{\mathrm{op}}, (Y')^{\mathrm{op}}, Z'}]{} & \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}((X')^{\mathrm{op}}\overline{\otimes}(Y')^{\mathrm{op}}, Z) \end{array}$$

Let $\beta : X \longrightarrow \overline{\mathrm{Hom}}(Y^{\mathrm{op}}, Z)$ be a chain map. For $x' \otimes y' + B_n(X' \otimes Y') \in (X'\overline{\otimes}Y')_n$, with $x' \in X'_k$ and $y' \in Y'_{n-k}$, we have:

$$\begin{aligned} &[\mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}\overline{\otimes}g^{\mathrm{op}}, h) \circ \Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}(\beta)]_n(x' \otimes y' + B_n(X' \otimes Y')) = \\ &= [h \circ \Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}(\beta) \circ (f\overline{\otimes}g)]_n(x' \otimes y' + B_n(X' \otimes Y')) \\ &= h_n \circ \Psi_{X^{\mathrm{op}}, Y^{\mathrm{op}}, Z}(\beta)_n(f_k(x') \otimes g_{n-k}(y') + B_n(X \otimes Y)) \\ &= (-1)^{n-k} h_n \circ \beta_k(f_k(x))_{n-k} \circ g_{n-k}(y'), \\ &(\Psi_{(X')^{\mathrm{op}}, (Y')^{\mathrm{op}}, Z'} \circ \mathrm{Hom}_{\mathbf{Ch}(_R\mathbf{Mod})}(f^{\mathrm{op}}, \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)))(\beta)_n(x' \otimes y' + B_n(X' \otimes Y')) = \\ &= (\Psi_{(X')^{\mathrm{op}}, (Y')^{\mathrm{op}}, Z'}(\overline{\mathrm{Hom}}(g^{\mathrm{op}}, h) \circ \beta \circ f))_n(x' \otimes y' + B_n(X' \otimes Y')) \\ &= (-1)^{n-k} \overline{\mathrm{Hom}}(g^{\mathrm{op}}, h)_k(\beta_k(f_k(x')))_{n-k}(y') \\ &= (-1)^{n-k} h_n \circ \beta_k(f_k(x'))_{n-k} \circ g_{n-k}(y'). \end{aligned}$$

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