Computing Ext using n-projective modules

Marco A. Pérez B.

Massachusetts Institute of Technology

Monday Topology Seminar

09 / 29 / 2014

Given a ring R and two left R-modules M and N, let's recall the "standard" way to compute the ith extension group $\mathrm{Ext}^i_R(M, N)$:

(1) Choose a (left) projective resolution of M:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

(2) Take the deleted complex:

$$\boldsymbol{P_\bullet} := \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

(3) Apply the contravariant functor $\mathrm{Hom}_R(-, N)$:

$$\mathrm{Hom}_R(\boldsymbol{P_\bullet}, N) := 0 \longrightarrow \mathrm{Hom}_R(P_0, N) \longrightarrow \mathrm{Hom}_R(P_1, N) \longrightarrow \cdots$$

$$\boxed{\mathrm{Ext}^i_R(M,N) := H^i(\mathrm{Hom}_R(P_\bullet, N))}$$

Question

Can we replace the class $\mathcal{P}_0$ of projective modules by other class of modules and still get $\mathrm{Ext}^i_R(M,N)$?

History

In 2004, James Gillespie proved that $\mathrm{Ext}^i_R(M, N)$ *can be computed by resolving* M *by flat modules and, simultaneously,* N *by cotorsion modules:*

$$0 \longrightarrow N \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$$

where each C^k *is a cotorsion module for every* $k \in \mathbb{Z}_{\geq 0}$*, i.e.* $\mathrm{Ext}^1_R(F, C^k) = 0$ *for every* F *flat.*

Question

What do $\mathcal{P}_0$ and the class $\mathcal{F}_0$ of flat modules have in common?

Answer

(1) *They are both resolving classes in ${}_R\mathbf{Mod}$.*

(2) *They are used to construct the class of cofibrant objects of a certain type of model structures on the category $\mathbf{Ch}(R)$ of chain complexes of left R-modules.*

Equivalently,

$$\boxed{\mathrm{Ext}^i_R(M,N) \cong \mathbf{Ch}(R)(P_\bullet, S^i(N))/\sim}$$

where $S^i(N)$ is the ith sphere chain complex centred at N:

$$\cdots \longrightarrow 0 \longrightarrow \underbrace{N}_{i\text{th place}} \longrightarrow 0 \longrightarrow \cdots$$

Proposition (J. Gillespie, 2004)

$\mathrm{Ext}^i_R(M,N) \cong \mathbf{Ch}(R)(\boldsymbol{Q},\boldsymbol{R})/\sim$, *where* $\boldsymbol{Q}$ *is a cofibrant replacement of* $S^0(M)$ *and* $\boldsymbol{R}$ *is a fibrant replacement of* $S^i(N)$ *in the flat model structure on* $\mathbf{Ch}(R)$.

Homological Algebra	**Homotopy Theory**
Cotorsion pairs in ${}_R\mathbf{Mod}$	Model Categories on $\mathbf{Ch}(R)$
Left resolutions of M by the left half of a cotorsion pair	Cofibrant replacement of $S^0(M)$ in a model category
Right resolutions of N by the right half of a cotorsion pair	Fibrant replacement of $S^i(N)$ in a model category

Let $\mathcal{P}_n$ be the class of n-projective modules M, i.e. $\mathrm{pd}(M) \leq n$.

(1) $\mathcal{P}_n$ is the left half of a *complete* cotorsion pair.

(2) There exists an *n-projective model structure* on $\mathbf{Ch}(R)$ where the cofibrant objects are "made up of" of n-projective modules.

Definition

Two classes $\mathcal{A}$ *and* $\mathcal{B}$ *of objects in an Abelian category* $\mathcal{C}$ *(say* ${}_R\mathbf{Mod}$ *or* $\mathbf{Ch}(R)$*) form a* ***cotorsion pair*** $(\mathcal{A}, \mathcal{B})$ *if:*

(1) $\mathcal{A} = {}^{\perp}\mathcal{B} := \{A \in \mathcal{C} : \mathrm{Ext}^1(A, B) = 0$ *for every* $B \in \mathcal{B}\}$*; and*

(2) $\mathcal{B} = \mathcal{A}^{\perp} := \{B \in \mathcal{C} : \mathrm{Ext}^1(A, B) = 0$ *for every* $A \in \mathcal{A}\}$.

The pair $(\mathcal{A}, \mathcal{B})$ *is* ***complete*** *if for every* $X \in \mathcal{C}$ *there exist short exact sequences*

$$0 \longrightarrow B \longrightarrow A \longrightarrow X \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow X \longrightarrow B' \longrightarrow A' \longrightarrow 0,$$

with $A, A' \in \mathcal{A}$ *and* $B, B' \in \mathcal{B}$.

Theorem (M. Hovey, 2002)

Let $\mathcal{C}$ be a bicomplete Abelian category and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{W}$ be three classes of objects in $\mathcal{C}$. If $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ are complete cotorsion pairs and $\mathcal{W}$ is thick, then there exists a unique model structure on $\mathcal{C}$ given by:

$$\begin{aligned}
\textit{cofibrations} &= \textit{monomorphisms with cokernel in } \mathcal{A}, \\
\textit{trivial cofibrations} &= \textit{monomorphisms with cokernel in } \mathcal{A} \cap \mathcal{W}, \\
\textit{fibrations} &= \textit{epimorphisms with kernel in } \mathcal{B}, \textit{ and} \\
\textit{trivial fibrations} &= \textit{epimorphisms with kernel in } \mathcal{B} \cap \mathcal{W}.
\end{aligned}$$

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in ${}_R\mathbf{Mod}$.

(1) $\widetilde{\mathcal{A}}$ denotes the class of complexes $A \in \mathbf{Ch}(R)$ such that:

(a) A is exact, and

(b) $Z_m(A) := \mathrm{Ker}(A_m \longrightarrow A_{m-1}) \in \mathcal{A}$ for every $m \in \mathbb{Z}$.

(2) $\widetilde{\mathcal{B}}$ denotes the class of complexes $B \in \mathbf{Ch}(R)$ such that:

(a) B is exact, and

(b) $Z_m(B) := \mathrm{Ker}(B_m \longrightarrow B_{m-1}) \in \mathcal{B}$ for every $m \in \mathbb{Z}$.

(3) $\mathrm{dg}\widetilde{\mathcal{A}}$ denotes the class of complexes $A \in \mathbf{Ch}(R)$ such that:

(a) $A_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$, and

(b) Every chain map $A \longrightarrow B$ is homotopic to zero, whenever $B \in \widetilde{\mathcal{B}}$.

(4) $\mathrm{dg}\widetilde{\mathcal{B}}$ denotes the class of complexes $B \in \mathbf{Ch}(R)$ such that:

(a) $B_m \in \mathcal{B}$ for every $m \in \mathbb{Z}$, and

(b) Every chain map $A \longrightarrow B$ is homotopic to zero, whenever $A \in \widetilde{\mathcal{A}}$.

Proposition (J. Gillespie, 2004)

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in ${}_R\mathbf{Mod}$. Then:

(1) $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ *and* $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ *are cotorsion pairs in* $\mathbf{Ch}(R)$.

(2) $\widetilde{\mathcal{A}} = \mathrm{dg}\widetilde{\mathcal{A}} \cap \mathcal{E}$ *and* $\widetilde{\mathcal{B}} = \mathrm{dg}\widetilde{\mathcal{B}} \cap \mathcal{E}$, *where* $\mathcal{E}$ *is the class of exact chain complexes, provided* $(\mathcal{A}, \mathcal{B})$ *is hereditary.*

Example (E. E. Enochs, 2001)

$(\mathcal{F}_0, (\mathcal{F}_0)^{\perp})$ *is a hereditary and complete cotorsion pair in* ${}_R\mathbf{Mod}$.

Example (J. Gillespie, 2004)

$(\mathrm{dg}\widetilde{\mathcal{F}_0}, \widetilde{(\mathcal{F}_0)^{\perp}})$ *and* $(\widetilde{\mathcal{F}_0}, \mathrm{dg}\widetilde{(\mathcal{F}_0)^{\perp}})$ *are complete cotorsion pairs in* $\mathbf{Ch}(R)$.

Theorem (-, 201N ∃! $4 \leq$ N ≤ 6)

There exists a unique model structure on $\mathbf{Ch}(R)$ defined as follows:

$$\begin{aligned}
\textit{cofibrations} &= \textit{injective chain maps with cokernel in } \mathrm{dg}\widetilde{\mathcal{P}_n}, \\
\textit{trivial cofibrations} &= \textit{injective chain maps with cokernel in } \widetilde{\mathcal{P}_n}, \\
\textit{fibrations} &= \textit{surjective chain maps with kernel in } \mathrm{dg}\widetilde{(\mathcal{P}_n)^{\perp}}, \\
\textit{trivial fibrations} &= \textit{surjective chain maps with kernel in } \widetilde{(\mathcal{P}_n)^{\perp}}, \\
\textit{weak equivalences} &= \textit{quasi-isomorphisms.}
\end{aligned}$$

(1) [E. E. Enochs et al, 2001]: $(\mathcal{P}_n, (\mathcal{P}_n)^{\perp})$ is a complete and hereditary cotorsion pair.

(2) $(\mathrm{dg}\widetilde{\mathcal{P}_n}, \widetilde{(\mathcal{P}_n)^{\perp}})$ and $(\widetilde{\mathcal{P}_n}, \mathrm{dg}\widetilde{(\mathcal{P}_n)^{\perp}})$ are cotorsion pairs such that $\widetilde{\mathcal{P}_n} = \mathrm{dg}\widetilde{\mathcal{P}_n} \cap \mathcal{E}$ and $\widetilde{(\mathcal{P}_n)^{\perp}} = \mathrm{dg}\widetilde{(\mathcal{P}_n)^{\perp}} \cap \mathcal{E}$.

(3) Are these two pairs complete?

Modules over ringoids

A **ringoid** is a small pre-additive category $\mathfrak{R}$.

A **(left) module** over $\mathfrak{R}$ is a functor $\mathfrak{R} \longrightarrow \mathbf{Ab}$.

$\mathbf{Mod}(\mathfrak{R})$ = category of left modules over $\mathfrak{R}$.

Example

Every ring R is a ringoid by setting $\mathrm{Ob}(R) = \{*\}$ and $R(*, *) = R$. Moreover, $\mathbf{Mod}(R) = {}_R\mathbf{Mod}$.

Example

Let $\mathfrak{C}$ be the ringoid generated by the infinite graph

$$\begin{tikzcd} \cdots \arrow[r] & 2 \arrow[loop above, "e_2"] \arrow[r, "\partial_2"] & 1 \arrow[loop above, "e_1"] \arrow[r, "\partial_1"] & 0 \arrow[loop above, "e_0"] \arrow[r, "\partial_0"] & -1 \arrow[loop above, "e_{-1}"] \arrow[r, "\partial_{-1}"] & -2 \arrow[loop above, "e_{-2}"] \arrow[r] & \cdots \end{tikzcd}$$

together with the relation $\partial_n \circ \partial_{n+1} = 0$*:*

$$\mathrm{Ob}(\mathfrak{C}) = \mathbb{Z},$$

$$\mathfrak{C}(i,j) = \begin{cases} \langle e_i \rangle := \{r \cdot e_i \, : r \in R\} & \textit{if } i = j, \\ \langle \partial_i \rangle := \{r \cdot \partial_i \, : r \in R\} & \textit{if } j = i+1, \\ 0 & \textit{otherwise.} \end{cases}$$

In this case, $\mathbf{Mod}(\mathfrak{C}) = \mathbf{Ch}(R)$.

Proposition

Let $\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R}))$ denote the class of n-projective modules over $\mathfrak{R}$. Then $(\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})), (\mathcal{P}_n(\mathbf{Mod}(\mathfrak{R})))^{\perp})$ is a complete and hereditary cotorsion pair.

For $\mathfrak{R} = \mathfrak{C}$: $\mathcal{P}_n(\mathbf{Mod}(\mathfrak{C})) = \widetilde{\mathcal{P}_n}$.

Corollary

$(\widetilde{\mathcal{P}_n}, \widetilde{\mathrm{dg}(\mathcal{P}_n)^{\perp}})$ *is a complete cotorsion pair.*

Proposition

Let $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$. If $\mathcal{W}$ is the left and right half of two complete cotorsion pairs $({}^{\perp}\mathcal{W}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{W}^{\perp})$, then

$$(\mathcal{A} \cap \mathcal{W}, \mathcal{B}) \text{ is complete iff } (\mathcal{A}, \mathcal{B} \cap \mathcal{W}) \text{ is complete.}$$

Corollary

$(\mathrm{dg}\widetilde{\mathcal{P}_n}, \widetilde{(\mathcal{P}_n)^{\perp}})$ *is a complete cotorsion pair.*

Consider

$$\mathbf{Ho}(\mathbf{Ch}(R)) = \mathbf{Ch}(R)[\mathcal{W}^{-1}]$$

where $\mathcal{W}$ = quasi-isomorphisms.

(1) [W. G. Dwyer & J. Spalinski]:

$$\mathrm{Ext}_R^{k-m}(M, N) \cong \mathbf{Ho}(\mathbf{Ch}(R))(S^m(M), S^k(N)).$$

In particular, $\mathrm{Ext}_R^i(M, N) \cong \mathbf{Ho}(\mathbf{Ch}(R))(S^0(N), S^i(N))$.

(2) [W. G. Dwyer & J. Spalinski]:

$$\mathbf{Ho}(\mathbf{Ch}(R))(S^m(M), S^k(N)) \cong \mathbf{Ch}(R)(\mathbf{Cof}, \mathbf{Fib})/\sim,$$
$$\mathbf{Cof} = \text{cofibrant replacement of } S^m(N),$$
$$\mathbf{Fib} = \text{fibrant replacement of } S^k(N).$$

Let $M, N \in_R \textbf{Mod}$:

$$\begin{tikzcd}
\cdots \arrow[r] & Q_1 \arrow[rr, "f_1"] \arrow[dr] & & \mathcal{P}_n \ni Q_0 \arrow[r, "f_0"] & M \arrow[r] & 0 \\
& & \mathrm{Ker}(f_0) \in (\mathcal{P}_n)^{\perp} \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where $Q_k \in \mathcal{P}_n$ and $\mathrm{Ker}(f_k) \in (\mathcal{P}_n)^{\perp}$ for every $k \geq 0$.

$$
\begin{tikzcd}
 & & \cdots \arrow[r] & Q_1 \arrow[r] \arrow[d, equal] & \mathrm{Ker}(f_0) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots & \in \widetilde{(\mathcal{P}_n)^{\perp}} \\
Q_\bullet \arrow[d, "p"] & = & \cdots \arrow[r] & Q_1 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d, "f_0"] & 0 \arrow[r] \arrow[d] & \cdots & \in \mathrm{dg}\widetilde{\mathcal{P}_n} \\
S^0(M) & = & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots &
\end{tikzcd}
$$

We have: p is a trivial fibration and ${\color{orange}Q_\bullet}$ is a cofibrant replacement of $S^0(M)$.

$$
\begin{tikzcd}
 & {\color{orange}Q_\bullet} \arrow[dr, two heads, "\sim", "p"'] & \\
0 \arrow[ur, hook] \arrow[rr] & & S^0(M)
\end{tikzcd}
$$

Similarly,

$$0 \longrightarrow N \longrightarrow R^0 \longrightarrow R^1 \longrightarrow \cdots$$

where $R^k \in (\mathcal{P}_n)^\perp$ and $\mathrm{Ker}(R^k \longrightarrow R^{k+1}) \in \mathcal{P}_n$ for every $k \geq 0$.

$$\begin{tikzcd}
 & R^\bullet \arrow[dr, two heads] & & & \Sigma^i(R^\bullet) \arrow[dr, two heads] & \\
S^0(N) \arrow[ur, hook, "\sim"] \arrow[rr] & & 0 \arrow[r, phantom, "\xrightarrow[\Sigma^i(-)]{\text{shift}}"] & S^i(N) \arrow[ur, hook, "\sim"] \arrow[rr] & & 0
\end{tikzcd}$$

where $\Sigma^i(R^\bullet)$ is a fibrant replacement of $S^i(N)$.

In particular,

$$\mathrm{Ext}^i_R(M, N) \cong \mathbf{Ch}(R)(Q_\bullet, \Sigma^i(R^\bullet))/\sim$$

(3) Given two chain complexes $X, Y \in \mathbf{Ch}(R)$,

$$\mathbf{Ch}(R)(X, \Sigma^i(Y))/\sim\ \cong H^i([X, Y]),$$

where $[X, Y]$ is the chain complex given by:

$$[X, Y]_m = \prod_{k \in \mathbb{Z}} \mathrm{Hom}_R(X_k, Y_{m+k}),$$

$$\begin{aligned} \partial_m^{[X,Y]} : [X, Y]_m &\longrightarrow [X, Y]_{m-1} \\ (f_k)_{k\in\mathbb{Z}} &\mapsto (\partial^Y_{m+k} \circ f_k - (-1)^m f_{k-1} \circ \partial^X_k)_{k\in\mathbb{Z}} \end{aligned}$$

Therefore,

$$\boxed{\mathrm{Ext}^i_R(M, N) \cong H^i([Q_\bullet, R^\bullet])}$$

$\mathfrak{FIM}$

Thank you for your attention!