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Covers by chain complexes with bounded Gorenstein-flat dimension

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Abstract

Covers by classes of chain complexes are currently one of the most important notions in homological algebra. Among the existing tools to construct covers, the theory of cotorsion pairs has turned out to be very useful. We show that every chain complex over a n-Gorenstein ring can be covered by the class of chain complexes with Gorenstein-flat dimension at most r, with r a non-negative integer smaller or equal than n, by proving that this class is the left half of a perfect cotorsion pair. This result represents an extension of the famous *Flat Cover Conjecture*, which was settled by E.E. Enochs in 2001.

Covers

Let $\mathbf{Ch}({}_R\mathbf{Mod})$ and $\mathbf{Ch}(\mathbf{Mod}_R)$ denote the categories of chain complexes of left and right R-modules, respectively.

Let $\mathcal{F}$ be a class of chain complexes in $\mathbf{Ch}({}_R\mathbf{Mod})$ and $X \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$. A chain map $f : F \longrightarrow X$ with $F \in \mathcal{F}$ is an ***$\mathcal{F}$-cover*** if the following two conditions are satisfied:

(1) For every chain map $f' : F' \longrightarrow X$ with $F' \in \mathcal{F}$, there exists a chain map $\varphi : F' \longrightarrow F$ such that $f \circ \varphi = f'$.

$$\begin{array}{ccc} F & \xrightarrow{f} & X \\ \exists!\ \varphi \uparrow & \nearrow_{f'} & \\ F' & & \end{array}$$

(2) If $F' = F$, the above diagram can only be completed by automorphisms of F.

If $f : F \longrightarrow X$ satisfies **(1)** but may be not **(2)**, then f is called an ***$\mathcal{F}$-pre-cover***.

An $\mathcal{F}$-pre-cover $f : F \longrightarrow X$ is said to be ***special*** if it is surjective and if $\mathrm{Ext}^1(F', \mathrm{Ker}(f)) = 0$ for every $F' \in \mathcal{F}$.

Envelopes, pre-envelopes and special pre-envelopes are defined dually.

Cotorsion pairs

Two classes $\mathcal{A}$ and $\mathcal{B}$ of chain complexes in $\mathbf{Ch}({}_R\mathbf{Mod})$ form a ***cotorsion pair*** $(\mathcal{A}, \mathcal{B})$ if:

(1) $A \in \mathcal{A}$ iff $\mathrm{Ext}^1(A, B) = 0$ for every $B \in \mathcal{B}$.

(2) $B \in \mathcal{B}$ iff $\mathrm{Ext}^1(A, B) = 0$ for every $A \in \mathcal{A}$.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be:

(a) ***Complete*** if every chain complex has a special $\mathcal{A}$-pre-cover and a special $\mathcal{B}$-pre-envelope.

(b) ***Perfect*** if every chain complex has an $\mathcal{A}$-cover and a $\mathcal{B}$-envelope.

(1) Theorem [R. Göbel & J. Trlifaf, 2006]

If $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair and $\mathcal{A}$ is closed under forming direct limits in $\mathbf{Ch}({}_R\mathbf{Mod})$, then $(\mathcal{A}, \mathcal{B})$ is perfect.

Two tensor products of chain complexes

Let $X \in \mathrm{Ob}(\mathbf{Ch}(\mathbf{Mod}_R))$ and $Y \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$.

- The ***standard tensor product*** of X and Y is the chain complex $X \otimes Y$ of Abelian groups given at each mth term by

$$(X \otimes Y)_m := \oplus_{k \in \mathbb{Z}} X_k \otimes_R Y_{m-k}$$

and whose boundary maps $\partial_m^{X \otimes Y} : (X \otimes Y)_m \longrightarrow (X \otimes Y)_{m-1}$ are defined as

$$x \otimes y \mapsto \partial_k^X(x) \otimes y + (-1)^k x \otimes \partial_{m-k}^Y(y)$$

at every $x \otimes y \in X_k \otimes_R Y_{m-k}$.

- The ***bar tensor product*** of X and Y is the chain complex $X \overline{\otimes} Y$ of Abelian groups given at each mth term by the quotient

$$(X \overline{\otimes} Y)_m := \frac{(X \otimes Y)_m}{B_m(X \otimes Y)}$$

where $B_m(X \otimes Y) := \mathrm{Im}(\partial_{m+1}^{X \otimes Y})$, and whose boundary maps $\partial_m^{X \overline{\otimes} Y} : (X \overline{\otimes} Y)_m \longrightarrow (X \overline{\otimes} Y)_{m-1}$ are defined as

$$x \otimes y + B_m(X \otimes Y) \mapsto \partial_k^X(x) \otimes y + B_{m-1}(X \otimes Y)$$

at every coset $x \otimes y + B_m(X \otimes Y) \in (X \overline{\otimes} Y)_m$ with $x \otimes y \in X_k \otimes_R Y_{m-k}$.

Let $\overline{\mathrm{Tor}}_i(-,-)$ denote the left derived functors of $-\overline{\otimes}-$.

Gorenstein-flat complexes

A chain complex E in $\mathbf{Ch}({}_R\mathbf{Mod})$ is said to be ***Gorenstein-flat*** is there is an exact sequence of flat chain complexes

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that $E = \mathrm{Ker}(F^0 \longrightarrow F^1)$ and such that the sequence

$$\cdots \longrightarrow I \overline{\otimes} F_1 \longrightarrow I \overline{\otimes} F_0 \longrightarrow I \overline{\otimes} F^0 \longrightarrow I \overline{\otimes} F^1 \longrightarrow \cdots$$

is exact for every injective chain complex I in $\mathbf{Ch}(\mathbf{Mod}_R)$.

Gorenstein-flat left R-modules are defined similarly with respect to the standard tensor product $\otimes_R$ of modules.

Let $\mathcal{GF}_0$ denote the class of Gorenstein-flat chain complexes.

Gorenstein rings

A ring R is an ***n-Gorenstein ring*** if it is left and right Noetherian and if its injective dimension (as a left R-module) is at most n.

If R is an n-Gorenstein ring, then $(\mathcal{GF}_0, (\mathcal{GF}_0)^\perp)$ is a complete cotorsion pair. Moreover, $\mathcal{GF}_0$ is closed under direct limits.

(2) Theorem [J. R. García Rozas, 1999]

If R is a commutative n-Gorenstein ring, then every chain complex has a Gorenstein-flat cover.

Gorenstein-flat dimension

Every chain complex X in $\mathbf{Ch}({}_R\mathbf{Mod})$ has a Gorenstein-flat resolution $\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow X \longrightarrow 0$.

Given an integer $0 \leq r \leq n$, we say that a chain complex X is ***Gorenstein-r-flat*** if there exists an exact sequence

$$0 \longrightarrow E_r \longrightarrow E_{r-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow X \longrightarrow 0$$

where each E_k is a Gorenstein-flat complex. In other words, X has a finite Gorenstein-flat resolution of length r. If r is the smallest integer for which such a sequence exists, we say that X has ***Gorenstein-flat dimension*** equal to r.

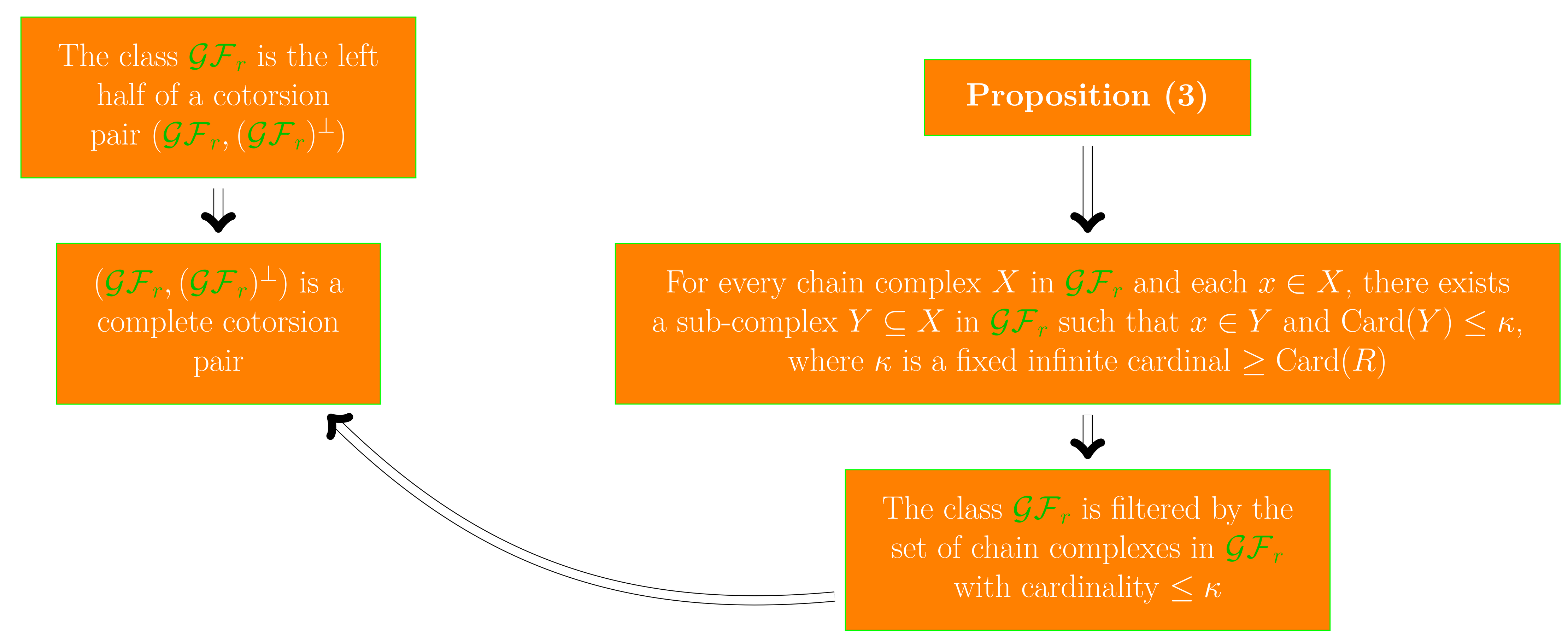
Gorenstein-r-flat left R-modules are defined similarly.

Let $\mathcal{GF}_r$ denote the class of Gorenstein-r-flat complexes.

(3) Proposition [properties of the Gorenstein-flat dimension]

Let R be an n-Gorenstein ring, and $\mathcal{W}$ denote the class of complexes in $\mathbf{Ch}(\mathbf{Mod}_R)$ with finite flat dimension. The following conditions are equivalent for every chain complex $X \in \mathrm{Ob}(\mathbf{Ch}({}_R\mathbf{Mod}))$ and every integer $0 \leq r \leq n$:

- X is a Gorenstein-r-flat complex.
- $\overline{\mathrm{Tor}}_{r+1}(W, X) = 0$ for every $W \in \mathcal{W}$.
- X_m is a Gorenstein-r-flat module for every $m \in \mathbb{Z}$.



For every chain complex X in $\mathbf{Ch}(\mathbf{Mod}_R)$, the functor $\overline{\mathrm{Tor}}(X, -)$ preserves direct limits, hence the class $\mathcal{GF}_r$ is closed under forming direct limits. The following theorem follows.

(4) Theorem [M. Pérez, 2012]

If R is a commutative n-Gorenstein ring, then every chain complex has a Gorenstein-r-flat cover, for each $0 \leq r \leq n$.

References

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