Gorenstein homological dimensions and abelian model structures

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Abstract

Let R be an associative ring with identity 1. We construct model structures from classes of modules with bounded Gorenstein homological dimensions. Specifically, if $\mathcal{GP}_r$ denotes the class of left R-modules with Gorenstein projective dimension at most r, then we get an abelian model structure on the category of left R-modules $R-\mathbf{Mod}$, such that $\mathcal{GP}_r$ is the class of cofibrant objects and $\mathcal{W}$ is the class of trivial objects, where $\mathcal{W}$ denotes the class of modules with finite projective dimension. Dually, we get an abelian model structure on $R-\mathbf{Mod}$ with the same trivial objects such that $\mathcal{GI}_r$, the class of modules with Gorenstein injective dimension at most r, forms the fibrant objects. These results are also valid for the category $\mathbf{Ch}(R-\mathbf{Mod})$ of chain complexes over $R-\mathbf{Mod}$.

Gorenstein homological dimensions

A left R-module M is called:

- ***Gorenstein projective*** if there exists a complex
$$\boldsymbol{P} = \cdots \to P^1 \to P^0 \to P_0 \to P_1 \to \cdots$$
of projective modules such that $M = \mathrm{Ker}(P_0 \to P_1)$ and $\mathrm{Hom}(\boldsymbol{P}, P)$ is exact for every projective module P. Gorenstein projective chain complexes are defined in the same way.
- ***Gorenstein injective*** if there exists a complex
$$\boldsymbol{I} = \cdots \to I^1 \to I^0 \to I_0 \to I_1 \to \cdots$$
of injective modules such that $M = \mathrm{Ker}(I_0 \to I_1)$ and $\mathrm{Hom}(I, \boldsymbol{I})$ is exact for every injective module I. Gorenstein injective chain complexes are defined in the same way.
- ***Gorenstein r-projective*** (or equiv., M has ***Gorenstein projective dimension*** r) if there exists an exact sequence
$$0 \to P_r \to P_{r-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$
where P_i is a Gorenstein projective module for all $0 \leq i \leq r$, and such that r is the smallest nonnegative integer satisfying this condition.
- ***Gorentein r-injective*** (or equiv., M has ***Gorenstein injective dimension*** r) if there exists an exact sequence
$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{r-1} \to I_r \to 0$$
where I_i is a Gorenstein injective module for all $0 \leq i \leq r$, and such that r is the smallest nonnegative integer satisfying this condition.

Cotorsion Pairs

- Two classes $\mathcal{C}$ and $\mathcal{F}$ of modules form a ***cotorsion pair*** $(\mathcal{C}, \mathcal{F})$ if $\mathcal{C} = {}^{\perp}\mathcal{F} = \{X : \mathrm{Ext}^1(X, F) = 0 \ \forall\, F \in \mathcal{F}\}$ and $\mathcal{F} = \mathcal{C}^{\perp} = \{Y : \mathrm{Ext}^1(C, Y) = 0 \ \forall\, C \in \mathcal{C}\}$.
- $(\mathcal{C}, \mathcal{F})$ is ***complete*** if for every left R-module X there exist short exact sequences $0 \longrightarrow F \longrightarrow C \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow X \longrightarrow F' \longrightarrow C' \longrightarrow 0$ with $C, C' \in \mathcal{C}$ & $F, F' \in \mathcal{F}$.
- $(\mathcal{C}, \mathcal{F})$ is ***cogenerated by a set*** $\mathcal{S}$ if $\mathcal{F} = \mathcal{S}^{\perp}$. Every cotorsion pair cogenerated by a set is complete [1].
- These definitions and results also apply to $\mathbf{Ch}(R-\mathbf{Mod})$.

- A ring R is an ***n-Gorenstein ring*** if it is left and right Noetherian and if its injective dimension (as a left R-module) is at most n. In this case, the classes $\mathcal{P}_n$ (modules with projective dimension at most n), $\mathcal{I}_n$ (modules with injective dimension at most n), and $\mathcal{W}$ coincide.

(1) Theorem [M. Hovey, 2002]

(a) Suppose R is an n-Gorenstein ring. Let $\mathcal{T}$ denote the set of all n-syzygies $T \in \Omega^n(R/I)$ as I runs over the left ideals of R. Then $\mathcal{T}$ cogenerates a cotorsion pair $(\mathcal{GP}_0, \mathcal{W})$.

(b) Let $\mathcal{S}$ denote the set of all i-syzygies $S \in \Omega^i(J)$, as i runs over the nonnegative integers and J runs over the indecomposable injective modules. Then $\mathcal{S}$ cogenerates a cotorsion pair $(\mathcal{W}, \mathcal{GI}_0)$.

- Given a left R-module M, the ***m-th sphere complex*** is defined as the chain complex $\cdots \to 0 \to M \to 0 \to \cdots$, where M appears at the m-th entry.
- Let $\mathcal{GP}_0$ and $\mathcal{GI}_0$ denote the classes of Gorenstein projective and Gorenstein injective chain complexes, respectively.

(2) Corollary [M. Pérez, 2012]

(a) Let R be an n-Gorenstein ring. The cotorsion pair $(\mathcal{GP}_0, \mathcal{W})$ is cogenerated by the set
$$\mathcal{T} = \{T \in \Omega^n(S^m(R/I)) : m \in \mathbb{Z} \text{ and } I \text{ is a left ideal of } R\} \cup \{\Sigma^k(S^0(R)) : k \in \mathbb{Z}\}.$$

(b) The cotorsion pair $(\mathcal{W}, \mathcal{GI}_0)$ is cogenerated by the set $\mathcal{S}$ of all $S \in \Omega^i(J)$ as i runs over the nonnegative integers and J runs over the indecomposable injective complexes.

(3) Theorem [M. Pérez, 2012]

(a) Suppose R is an n-Gorenstein ring and $0 < r \leq n$. Suppose κ is an infinite cardinal such that $\kappa > \mathrm{Card}(R)$. Then $(\mathcal{GP}_r, (\mathcal{GP}_r)^{\perp})$ is a cotorsion pair cogenerated by the set $\mathcal{T} \cup \mathcal{P}_r^{\leq\kappa}$, where $\mathcal{P}_r^{\leq\kappa} := \{M \in \mathcal{P}_r : \mathrm{Card}(M) \leq \kappa\}$.

(b) Let $\mathcal{S}_r$ denote the set of all $S \in \Omega^i(J)$, as i runs over the integers $\geq r$ and J runs over the indecomposable injective modules. Then $({}^{\perp}(\mathcal{GI}_r), \mathcal{GI}_r)$ is cogenerated by the set $\mathcal{S}_r$.

The same results are also valid in $\mathbf{Ch}(R-\mathbf{Mod})$.

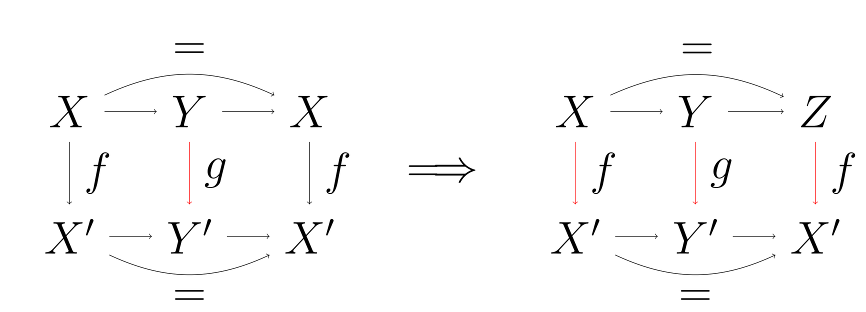
Model Categories

- Let $\mathcal{A}$ be a bicomplete category. A ***model structure*** on $\mathcal{A}$ is formed by three classes of morphisms $\mathcal{C}of$, $\mathcal{F}ib$ and $\mathcal{W}eak$ called **cofibrations**, **fibrations** and **weak equivalences**, respectively, satisfying:

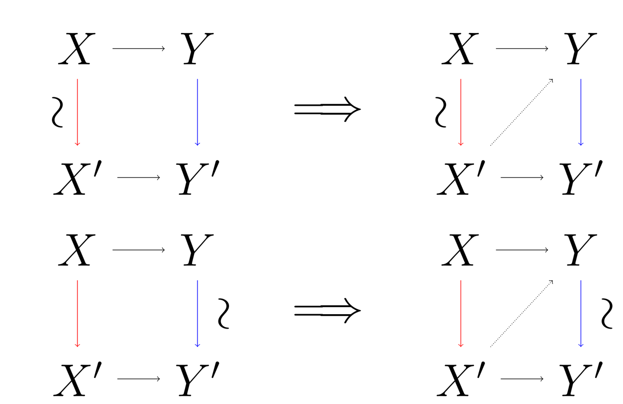
(a) 3 x 2 for weak equivalences: For all composable morphisms f and g in $\mathcal{A}$, if two of the three morphisms f, g and $g \circ f$ are weak equivalences, then so is the third.

(b) Factorization property: For each map f in $\mathcal{A}$, there exist $\alpha \in \mathcal{C}of$, $\gamma \in \mathcal{C}of \cap \mathcal{W}eak$, $\delta \in \mathcal{F}ib$ and $\beta \in \mathcal{F}ib \cap \mathcal{W}eak$ such that f can by written as $f = \beta \circ \alpha = \delta \circ \gamma$.

(c) $\mathcal{C}of$, $\mathcal{F}ib$ and $\mathcal{W}eak$ are closed under retractions



(d) Left lifting property



- An object X is **cofibrant** if $(0 \to X) \in \mathcal{C}of$; **fibrant** if $(X \to 0) \in \mathcal{F}ib$; and **trivial** if $(0 \to X) \in \mathcal{W}eak$.
- A model structure $(\mathcal{C}of, \mathcal{F}ib, \mathcal{W}eak)$ is ***abelian*** if
  - (a) $f \in \mathcal{C}of$ iff f is monic with cofibrant cokernel.
  - (b) $g \in \mathcal{F}ib$ iff g is epic with fibrant kernel.

Gorenstein model structures

(4) Hovey's Criterion [2002]

Let $\mathcal{A}$ be a bicomplete abelian category with enough projective and injective objects. If $(\mathcal{C}, \mathcal{F} \cap \mathcal{E})$ and $(\mathcal{C} \cap \mathcal{E}, \mathcal{F})$ are complete cotorsion pairs in $\mathcal{A}$, and the class $\mathcal{E}$ is thick, then there exists a unique abelian model structure on $\mathcal{A}$ such that $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{E}$ are the classes of cofibrant objects, fibrant objects, and trivial objects, respectively.

- In [5], it is proven that if R is an n-Gorenstein ring, then $\mathcal{P}_r = \mathcal{GP}_r \cap \mathcal{W}$, $(\mathcal{GP}_r)^{\perp} = (\mathcal{P}_r)^{\perp} \cap \mathcal{W}$, $\mathcal{I}_r = \mathcal{GI}_r \cap \mathcal{W}$ and ${}^{\perp}(\mathcal{GI}_r) = {}^{\perp}(\mathcal{I}_r) \cap \mathcal{W}$, for every $0 \leq r \leq n$.

(5) Corollary [M. Pérez, 2012]

(a) There exists a unique abelian model structure on $R-\mathbf{Mod}$ such that $\mathcal{GP}_r$ is the class of cofibrant objects, $(\mathcal{P}_r)^{\perp}$ is the class of fibrant objects, and $\mathcal{W}$ is the class of trivial objects.

(b) There exists a unique abelian model structure on $R-\mathbf{Mod}$ such that $\mathcal{GI}_r$ is the class of fibrant objects, ${}^{\perp}(\mathcal{I}_r)$ is the class of cofibrant objects, and $\mathcal{W}$ is the class of trivial objects.

The same results are also valid in $\mathbf{Ch}(R-\mathbf{Mod})$.

Gorenstein model structures and graded rings

- A chain complex X in $\mathbf{Ch}(R-\mathbf{Mod})$ is said to be ***differential graded r-projective*** if $X_m \in \mathcal{P}_r$ for every $m \in \mathbb{Z}$, and if every chain map $X \to Y$ is null homotopic, whenever Y is an exact complex satisfying $\mathrm{Ext}^1(Z, Y) = 0$, for every complex Z with projective dimension at most r.
- A chain complex Y in $\mathbf{Ch}(R-\mathbf{Mod})$ is said to be ***differential graded r-injective*** if $Y_m \in \mathcal{I}_r$ for every $m \in \mathbb{Z}$, and if every chain map $X \to Y$ is null homotopic, whenever X is an exact complex satisfying $\mathrm{Ext}^1(X, Z) = 0$, for every complex Z with injective dimension at most r.
- The quotient ring $A := R[x]/(x^2)$ is a $\mathbb{Z}$-graded ring. There is an invertible functor $\Phi : A-\mathbf{Mod} \to \mathbf{Ch}(R-\mathbf{Mod})$ such that every $\mathbb{Z}$-graded A-module $M = (M_m)_{m \in \mathbb{Z}}$ is mapped into the chain complex
$$\Phi(M) = \cdots \to M_{m+1} \xrightarrow{\partial_{m+1}} M_m \xrightarrow{\partial_m} M_{m-1} \to \cdots$$
where the boundary maps ∂_m are given by $y \mapsto x \cdot y$.

The following result is a generalization of [3, Prop. 3.6 & 3.8]:

(6) Theorem [M. Pérez, 2012]

The functor $\Phi^{-1} : \mathbf{Ch}(R-\mathbf{Mod}) \to A-\mathbf{Mod}$ maps:

(a) differential graded r-projective complexes into Gorenstein r-projective A-modules, and

(b) differential graded r-injective complexes into Gorenstein r-injective A-modules.

If R is an n-Gorenstein ring, then the functor $\Phi : A-\mathbf{Mod} \to \mathbf{Ch}(R-\mathbf{Mod})$ maps:

(c) Gorenstein r-projective A-modules into differential graded r-projective complexes, and

(d) Gorenstein r-injective A-modules into differential graded r-injective complexes.

References

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