Special $n$-projective precovers of modules over ringoids

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Let $C$ be an Abelian category with enough projective objects.

**Definition ($n$-projective objects)**

Given $n \in \mathbb{Z}_{\geq 0}$. An object $X$ of $C$ is called $n$-projective if it has projective dimension at most $n$.

$$\mathcal{P}_{\leq n}(C) := \{ X \in \text{Ob}(C) : \text{pd}(X) \leq n \}.$$
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**Question**

Is $\mathcal{P}_{\leq n}(C)$ a special precovering class?
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**Question**

Is $\mathcal{P}_{\leq n}(C)$ a special precovering class?

In other words,

**Question**

Is $\mathcal{P}_{\leq n}(C)$ the left half of a complete cotorsion pair?
We tackle the previous question:

(1) for **two** particular choices of $C$, namely

$$R\text{Mod} := \text{the category of left } R\text{-modules},$$

$$\text{Ch}(R\text{Mod}) := \text{the category of complexes of (left) } R\text{-modules},$$

and,

(2) using **one** strategy.
We tackle the previous question:

(1) for two particular choices of $C$, namely

$$R\text{Mod} := \text{the category of left } R\text{-modules},$$
$$\text{Ch}(R\text{Mod}) := \text{the category of complexes of (left) } R\text{-modules},$$

and,

(2) using one strategy. Namely,

we write $R\text{Mod}$ and $\text{Ch}(R\text{Mod})$ as categories of left modules over certain ringoids $\mathcal{R}$, and give a positive answer for the previous question in the category $\text{Mod}(\mathcal{R})$ of modules over $\mathcal{R}$.
**Notation**

\[ \mathcal{P}_{\leq n}(R\text{Mod}) = \mathcal{P}_{\leq n} = n\text{-projective } R\text{-modules.} \]


Let \( R \) be a ring, \( \kappa > \text{Card}(R) \) and \( L \in \mathcal{P}_{\leq n} \). If \( x \in L \), then there exists a submodule \( L' \subseteq L \) such that \( x \in L' \), \( \text{Card}(L') \leq \kappa \), and \( L', L/L' \in \mathcal{P}_{\leq n} \).


\( (\mathcal{P}_{\leq n}, (\mathcal{P}_{\leq n})^\perp) \) is a complete and hereditary cotorsion pair in \( R\text{Mod} \).

In particular, \( \mathcal{P}_{\leq n} \) is a special precovering class.
(1) Introduce the category $\text{Mod}(\mathcal{R})$ of modules over a ringoid $\mathcal{R}$.

(2) Encode the categories $\_R\text{Mod}$ and $\text{Ch}(\_R\text{Mod})$ as modules over certain ringoids.

(3) Show that $\mathcal{P}_{\leq n}(\text{Mod}(\mathcal{R}))$ is a special precovering class in $\text{Mod}(\mathcal{R})$.

(4) Applications: $n$-Projective modules and some cotorsion pairs in chain complexes.
Model structures on $\text{Ch}(\_R\text{Mod})$. 
Some references

This talk presents a part of the results appearing in:


and inspired in the following works:


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Marco A. Pérez (maperez@mit.edu) Special $n$-projective precovers of modules over ringoids
What are ringoids and modules over them?

**Definition (ringoids)**

A **ringoid** $\mathcal{R}$ is a small preadditive category.

**Definition (modules over a ringoid)**

Let $\text{Ab}$ denote the category of Abelian groups. A **(left) $\mathcal{R}$-module** is given by a covariant functor $M: \mathcal{R} \to \text{Ab}$.

**Definition (morphisms between ringoids)**

Given two modules over $\mathcal{R}$, say $M, N: \mathcal{R} \to \text{Ab}$. A **morphism** from $M$ to $N$ is a natural transformation $M \Rightarrow N$.

Let $\text{Mod}(\mathcal{R})$ denote the Abelian category of left modules over $\mathcal{R}$.
1. Introduce the category $\text{Mod}(\mathcal{R})$ of modules over a ringoid $\mathcal{R}$.

2. Encode the categories $R\text{Mod}$ and $\text{Ch}(R\text{Mod})$ as modules over certain ringoids.

3. Show that $\mathcal{P}_{\leq n}(\text{Mod}(\mathcal{R}))$ is a special precovering class in $\text{Mod}(\mathcal{R})$.

   Model structures on $\text{Ch}(R\text{Mod})$. 

Marco A. Pérez (maperez@mit.edu) Special $n$-projective precovers of modules over ringoids
Some examples

Example (Modules over a ring)

Let $R$ be a ring.

- The category $\mathcal{M}$ given by the collections:
  
  \[
  \text{Ob}(\mathcal{M}) := \{\ast\}, \text{ and} \\
  \text{Hom}_{\mathcal{M}}(\ast, \ast) := R,
  \]

  define a ringoid.

- In this case, we have $\text{Mod}(\mathcal{M}) = R\text{Mod}$.
Some examples

Example (Chain complexes over a ring)

Consider the infinite graph

\[ \cdots \rightarrow e_2 \circlearrowleft \rightarrow e_1 \rightarrow e_0 \rightarrow e_{-1} \rightarrow e_{-2} \circlearrowleft \rightarrow \cdots \]

\[ \cdots \rightarrow 2 \rightarrow 1 \rightarrow 0 \rightarrow -1 \rightarrow -2 \rightarrow \cdots \]

together with the following relations for every \( n \in \mathbb{Z} \):

\[ \partial_n \circ \partial_{n+1} = 0, \]
\[ \partial_n \circ e_n = \partial_n, \]
\[ e_{n-1} \circ \partial_n = \partial_n, \]
\[ e_n \circ e_n = e_n. \]
Some examples

Example (Chain complexes over a ring)

Let $R$ be a ring. Set $\mathcal{C}$ as the ringoid defined by the following collections of objects and morphisms:

$$\text{Ob}(\mathcal{C}) = \mathbb{Z},$$

$$\text{Hom}_{\mathcal{C}}(n, m) = \begin{cases} 
\langle e_n \rangle := \{k \cdot e_n : k \in \mathbb{Z}\} & \text{if } n = m, \\
\langle \partial_n \rangle := \{k \cdot \partial_n : k \in \mathbb{Z}\} & \text{if } m = n - 1, \\
0 & \text{otherwise.}
\end{cases}$$

In this case, $\text{Mod}(\mathcal{C}) = \text{Ch}(\text{Ab}).$
Some examples

**Example (Chain complexes over a ring)**

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0 & \text{otherwise.}
\end{cases}$$

In this case, $\text{Mod}(\mathcal{C}) = \text{Ch}(\text{Ab})$. And

$$\text{Ch}(R \text{Mod}) = \text{Func}(\mathcal{C}, R \text{Mod}) = \text{Func}(\mathcal{C}, \text{Func}(M, \text{Ab}))$$

$$\cong \text{Func}(\mathcal{C} \otimes M, \text{Ab}) = \text{Mod}(\mathcal{C} \otimes M).$$
Example (A very easy but important example)

Let $\mathcal{R}$ be a ringoid. For every $a \in \text{Ob}(\mathcal{R})$, $\text{Hom}_\mathcal{R}(a, -) : \mathcal{R} \to \text{Ab}$ is a left module over $\mathcal{R}$. 

Definition (Free modules over a ringoid)

An $\mathcal{R}$-module $F$ is free if it is isomorphic (as objects in $\text{Mod}(\mathcal{R})$) to a coproduct of $\mathcal{R}$-modules $\text{Hom}_\mathcal{R}(a_i, -)$ for a family $\{a_i\}_{i \in I}$ of objects of $\mathcal{R}$:

$$M \bigoplus_{i \in I} \text{Hom}_\mathcal{R}(a_i, -).$$
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Definition (Free modules over a ringoid)

An $\mathcal{R}$-module $F$ is \textbf{free} if it is isomorphic (as objects in $\text{Mod}(\mathcal{R})$) to a coproduct of $\mathcal{R}$-modules $\text{Hom}_\mathcal{R}(a_i, -)$ for a family $\{a_i\}_{i \in I}$ of objects of $\mathcal{R}$:

$$M \cong \bigoplus_{i \in I} \text{Hom}_\mathcal{R}(a_i, -).$$
(1) Introduce the category Mod(\mathcal{R}) of modules over a ringoid \mathcal{R}.

(2) Encode the categories \textit{R}Mod and Ch(\textit{R}Mod) as modules over certain ringoids.

(3) Show that \mathcal{P}_{\leq n}(\text{Mod(}\mathcal{R}\text{)}) is a special precovering class in \text{Mod(}\mathcal{R}\text{)}.

(4) Applications: \textit{n}-Projective modules and some cotorsion pairs in chain complexes.
Model structures on Ch(\textit{R}Mod).
Zig-zag procedure in $\text{Mod}(\mathcal{R})$

Recall:

**Main Theorem**

The class $\mathcal{P}_{\leq n}(\text{Mod}(\mathcal{R}))$ of $n$-projective modules over a ringoid $\mathcal{R}$ is special precovering.
Recall:

Main Theorem

The class $\mathcal{P}_{\leq n}(\text{Mod}(\mathcal{R}))$ of $n$-projective modules over a ringoid $\mathcal{R}$ is special precovering.

For this, we need:

Main Lemma

Let $M$ be a $n$-projective $\mathcal{R}$-module. Then for every “homogeneous” element $x \in M(a)$ there exists a “small submodule” $N \hookrightarrow M$ such that:

1. $x \in N(a)$.
2. The $\mathcal{R}$-modules $N$ and $M/N$ are $n$-projective.
What do we mean by “elements” of an $R$-module?

Definition (Homogeneous elements of a module)

If $M$ is an $R$-module, we say that $x$ is a **homogeneous element of $M$ of grade** $a$ if $x \in M(a)$, with $a \in \text{Ob}(R)$. This will be denoted by $|x| = a$. 

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What does it mean that \( N \) is a “submodule” of \( M \)?

For every \( \mathcal{R} \)-module \( M: \mathcal{R} \to \text{Ab} \) and for every pair \( a, b \in \text{Ob}(\mathcal{R}) \), there is an induced multiplication

\[
\text{Hom}_{\mathcal{R}}(a, b) \times M(a) \to M(b)
\]

\[
(r, x) \mapsto r \cdot x := M(r)(x).
\]
What does it mean that $N$ is a “submodule” of $M$?

For every $\mathcal{R}$-module $M: \mathcal{R} \rightarrow \text{Ab}$ and for every pair $a, b \in \text{Ob}(\mathcal{R})$, there is an induced multiplication

$$\text{Hom}_{\mathcal{R}}(a, b) \times M(a) \rightarrow M(b)$$

$$(r, x) \mapsto r \cdot x := M(r)(x).$$

**Definition (Submodules over a ringoid)**

If $M$ is a left $\mathcal{R}$-module, a family $N = \{N(a) : a \in \text{Ob}(\mathcal{R})\}$ of subgroups $N(a) \subseteq M(a)$ is a **submodule of $M$** if

$$x \in N(a) \implies r \cdot x \in N(b), \text{ for every } r \in \text{Hom}_{\mathcal{R}}(a, b).$$
And what does it mean that an $\mathcal{R}$-module is “small”?

**Definition (Small $\mathcal{R}$-modules)**

Let $\kappa$ be an infinite cardinal such that:

$$\kappa > \text{Card}(\text{Hom}_\mathcal{R}(a, b)), \ \forall a, b \in \text{Ob}(\mathcal{R}).$$

An $\mathcal{R}$-module $M$ is $\kappa$-**small** if $\text{Card}(M(a)) \leq \kappa$, for every $a \in \text{Ob}(\mathcal{R})$. 
Definition (admissible elements)

A linear combination of homogenous elements \( y = \sum_{i \in I} r_i \cdot x_i \) is **admissible** if \( y \) is homogenous and \( r_i \in \text{Hom}_R(|x_i|, |y|) \) for every \( i \in I \).
Definition (admissible elements)

A linear combination of homogenous elements \( y = \sum_{i \in I} r_i \cdot x_i \) is **admissible** if \( y \) is homogenous and \( r_i \in \text{Hom}_R(|x_i|, |y|) \) for every \( i \in I \).

Definition (basis)

A family \( \{x_i\}_{i \in I} \) of homogenous elements of an \( R \)-module \( M \) is a **basis** of \( M \) if every homogenous element \( x \in M \) can be written uniquely as an admissible linear combination \( x = \sum_{i \in I} r_i \cdot x_i \).
An \( R \)-module is free iff it admits a basis

\[ \Downarrow \]

\textbf{Main Lemma} \quad \Rightarrow \quad \text{Every } n\text{-projective } R\text{-module is \textbf{filtered} by the set of small } n\text{-projective } R\text{-modules}

\[ \Downarrow \]

\textbf{Main Theorem} \quad \Leftarrow \quad \mathcal{P}_{\leq n}(\text{Mod}(R)) \text{ is the left half of a cotorsion pair cogenerated by the set of small } n\text{-projective } R\text{-modules}
Applications to model categories

Notation and facts

(1) If $\mathcal{X}$ is a class of objects in an Abelian category $\mathcal{C}$, $\tilde{\mathcal{X}}$ denotes the class of exact chain complexes $X$ with $\mathbb{Z}_m(X) \in \mathcal{X}$, for every $m \in \mathbb{Z}$.

(2) $\mathcal{P}_{\leq n}(\text{Ch}(\mathcal{R\text{-Mod}})) = \tilde{\mathcal{P}}_{\leq n}$.

(3) If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{C}$, then

- $\text{dg} \tilde{\mathcal{A}}$ is the class of chain complexes $X$ such that $X_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$, and such that $X \to B$ is null homotopic for every $B \in \tilde{\mathcal{B}}$.
- $\text{dg} \tilde{\mathcal{B}}$ is defined dually.
Applications to model categories

\[(P_{\leq n}(\text{Mod}(R)), P_{\leq n}(\text{Mod}(R))^\perp)\]

is a complete cotorsion pair in \(\text{Mod}(R)\)

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Applications to model categories

$(\mathcal{P}_{\leq n}(\text{Mod}(\mathcal{R})), \mathcal{P}_{\leq n}(\text{Mod}(\mathcal{R}))^\perp)$

is a complete cotorsion pair in $\text{Mod}(\mathcal{R})$

$\mathcal{R} = \mathcal{M}$

[Enochs et al., 2001]

$(\mathcal{P}_{\leq n}, (\mathcal{P}_{\leq n})^\perp)$ is a complete cotorsion pair
Applications to model categories

\( \mathcal{R} = \mathcal{M} \)

[Enochs et al., 2001]

\((\mathcal{P}_{\leq n}, (\mathcal{P}_{\leq n})^\perp)\) is a complete cotorsion pair

\( \mathcal{R} = \mathcal{C} \otimes \mathcal{M} \)

\((\overline{\mathcal{P}}_{\leq n}, (\overline{\mathcal{P}}_{\leq n})^\perp)\) is a complete cotorsion pair
Applications to model categories

Proposition (J. Gillespie)

Suppose $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair in an Abelian category $C$. If $C$ has enough injective objects, then

$$\widetilde{\mathcal{A}} = \text{dg} \widetilde{\mathcal{A}} \cap \text{Ex}.$$
Applications to model categories

\[ \mathcal{R} = \mathcal{M} \]

\[ (\mathcal{P}_{\leq n}(\text{Mod}(\mathcal{R})), \mathcal{P}_{\leq n}(\text{Mod}(\mathcal{R}))^\perp) \]
is a complete cotorsion pair in \( \text{Mod}(\mathcal{R}) \)

\[ \mathcal{R} = \mathcal{C} \otimes \mathcal{M} \]

\[ (\tilde{\mathcal{P}}_{\leq n}, (\tilde{\mathcal{P}}_{\leq n})^\perp) \]
is a complete cotorsion pair

\[ (\text{dg}\tilde{\mathcal{P}}_{\leq n} \cap \text{Ex}, \text{dg}(\tilde{\mathcal{P}}_{\leq n})^\perp) \]
is a complete cotorsion pair

[Enochs et al., 2001]

\((\mathcal{P}_{\leq n}, (\mathcal{P}_{\leq n})^\perp)\) is a complete cotorsion pair

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Applications to model categories

\( R = M \) 🔄

[Enochs et al., 2001]
\((P_{\leq n}, (P_{\leq n})^\perp)\) is a complete cotorsion pair

\[
(P_{\leq n}(\text{Mod}(R)), P_{\leq n}(\text{Mod}(R))^\perp)
\]
is a complete cotorsion pair in \( \text{Mod}(R) \)

\[
R = C \otimes M
\]

[Enochs et al., 2001]
\((P_{\leq n}, (P_{\leq n})^\perp)\) is a complete cotorsion pair

[Enochs et al., 2001]
\((dgP_{\leq n}, dg(P_{\leq n})^\perp \cap \text{Ex})\) is a (complete) cotorsion pair

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Applications to model categories

Theorem (Hovey’s correspondence - But just one part of it)

Let $Q$, $R$ and $W$ be three classes of objects in a bicomplete Abelian category $C$ such that:

- $W$ is **thick**.
- $(Q \cap W, R)$ and $(Q, R \cap W)$ are complete cotorsion pair.

Then there exists a unique Abelian model structure on $C$ such that:

- $Q$ is the class of cofibrant objects.
- $R$ is the class of fibrant objects.
- $W$ is the class of trivial objects.

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\( \mathcal{R} = \mathcal{M} \)

[Enochs et al., 2001]

\((\mathcal{P}_{\leq n}, (\mathcal{P}_{\leq n})^\perp)\) is a complete cotorsion pair

\( R = C \otimes M \)

(Gillespie, 2008)

\((dg\mathcal{P}_{\leq n}, dg(\mathcal{P}_{\leq n})^\perp \cap Ex)\) is a (complete) cotorsion pair

Hovey’s correspondence

There is a unique model structure on \( Ch(R\text{Mod}) \) such that:

\( dg\mathcal{P}_{\leq n} \) is the class of cof. objects
\( dg(\mathcal{P}_{\leq n})^\perp \) is the class of fib. objects
\( Ex \) is the class of trivial objects

Hovey’s correspondence

Special \( n \)-projective precovers of modules over ringoids
Proposition (Compatibility and completeness)

Let $C$ be an Abelian category with enough projective and injective objects, and let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{W}$ be three classes of objects in $C$ such that $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ are cotorsion pairs. If $(\mathcal{W}, \mathcal{W}^\bot)$ and $(\mathcal{W}^\bot, \mathcal{W})$ are complete cotorsion pairs in $C$, then $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ is complete if, and only if, $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ is.
Completeness of \((\text{dg}P_{\leq n}, \text{dg}(P_{\leq n})^\perp \cap \text{Ex})\)

Example (of \(W\))

- **García Rozas**: The class \(\text{Ex}\) of exact chain complexes: \((\text{DG-projectives}, \text{Ex})\) and \((\text{Ex}, \text{DG-injectives})\) are complete cotorsion pairs in \(\text{Ch}(\text{RMod})\).
Example (of $W$)

- **García Rozas:** The class $\text{Ex}$ of exact chain complexes: $(\text{DG-projectives, } \text{Ex})$ and $(\text{Ex}, \text{DG-injectives})$ are complete cotorsion pairs in $\text{Ch}(R\text{Mod})$.

- **Enochs & Jenda:** The class $\mathcal{U} = \{ U \in \text{Ob}(R\text{Mod}) : \text{id}(U) < \infty \}$:
  
  $(\text{Gorenstein-projectives, } \mathcal{U})$ and $(\mathcal{U}, \text{Gorenstein-injectives})$
  
  are complete cotorsion pairs in $R\text{Mod}$, provided $R$ is a Gorenstein ring.
Completeness of \((\text{dgP}_{\leq n}, \text{dg}(\text{P}_{\leq n})^\perp \cap \text{Ex})\)

**Example (of \(\mathcal{W}\))**

- **García Rozas:** The class \(\text{Ex}\) of exact chain complexes: \((\text{DG-projectives, Ex})\) and \((\text{Ex, DG-injectives})\) are complete cotorsion pairs in \(\text{Ch}(\text{RMod})\).

- **Enochs & Jenda:** The class \(\mathcal{U} = \{U \in \text{Ob}(\text{RMod}) : \text{id}(U) < \infty\}\): \((\text{Gorenstein-projectives, } \mathcal{U})\) and \((\mathcal{U}, \text{Gorenstein-injectives})\) are complete cotorsion pairs in \(\text{RMod}\), provided \(\text{R}\) is a Gorenstein ring.

- **Gillespie:** The class \(\mathcal{V} = \{V \in \text{Ob}(\text{RMod}) : \text{FP-id}(V) < \infty\}\): \((\text{Ding-projectives, } \mathcal{V})\) and \((\mathcal{V}, \text{Ding-injectives})\) are complete cotorsion pair in \(\text{RMod}\), provided \(\text{R}\) is a Ding-Chen ring.
Completeness of \((\text{dg} \overrightarrow{P}_{\leq n}, \text{dg} (\overrightarrow{P}_{\leq n})^\perp \cap \text{Ex})\)

Knowing that \((\text{dg} \overrightarrow{P}_{\leq n}, \text{dg} (\overrightarrow{P}_{\leq n})^\perp \cap \text{Ex})\) and \((\text{dg} \overrightarrow{P}_{\leq n} \cap \text{Ex}, \text{dg} (\overrightarrow{P}_{\leq n})^\perp)\) are cotorsion pairs, where the latter is complete, we have:

**Corollary**

\((\text{dg} \overrightarrow{P}_{\leq n}, \text{dg} (\overrightarrow{P}_{\leq n})^\perp \cap \text{Ex})\) is a complete cotorsion pair.
¡Gracias por su atención!

These slides, plus thesis and papers (containing details omitted here) available from:

http://web.mit.edu/maperez/www/