Special *n*-projective precovers of modules over ringoids

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Introduction and motivation

Let $\mathcal{C}$ be an Abelian category with enough projective objects.

Definition (n-projective objects)

Given $n \in \mathbb{Z}_{\geq 0}$. An object X of $\mathcal{C}$ is called ***n*-projective** if it has projective dimension at most n.

$$\mathcal{P}_{\leq n}(\mathcal{C}) := \{X \in \mathrm{Ob}(\mathcal{C}) : \mathrm{pd}(X) \leq n\}.$$

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Question

Is $\mathcal{P}_{\leq n}(\mathcal{C})$ a **special precovering** *class?*

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In other words,

Question

Is $\mathcal{P}_{\leq n}(\mathcal{C})$ the left half of a ***complete*** *cotorsion pair?*

Introduction and motivation

We tackle the previous question:

(1) for **two** particular choices of C, namely

$$
\begin{aligned}
{}_R\mathrm{Mod} &:= \text{the category of left } R\text{-modules},\\
\mathrm{Ch}({}_R\mathrm{Mod}) &:= \text{the category of complexes of (left) } R\text{-modules},
\end{aligned}
$$

and,

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and,

(2) using **one** strategy. Namely,

we write ${}_R\mathrm{Mod}$ *and* $\mathrm{Ch}({}_R\mathrm{Mod})$ *as categories of left modules over certain ringoids* $\mathfrak{R}$*, and give a positive answer for the previous question in the category* $\mathrm{Mod}(\mathfrak{R})$ *of modules over* $\mathfrak{R}$*.*

Some history

Notation

$\mathcal{P}_{\leq n}({}_R\mathrm{Mod}) = \mathcal{P}_{\leq n}$ = *n-projective R-modules.*

Lemma (2001. Enochs, Yenda, Aldrich, Oyonarte)

Let R be a ring, $\kappa > \mathrm{Card}(R)$ and $L \in \mathcal{P}_{\leq n}$. If $x \in L$, then there exists a submodule $L' \subseteq L$ such that $x \in L'$, $\mathrm{Card}(L') \leq \kappa$, and $L', L/L' \in \mathcal{P}_{\leq n}$.

Theorem (2001. Enochs, Jenda, Aldrich, Oyonarte)

$(\mathcal{P}_{\leq n}, (\mathcal{P}_{\leq n})^{\perp})$ is a complete and hereditary cotorsion pair in ${}_R\mathrm{Mod}$. In particular, $\mathcal{P}_{\leq n}$ is a special precovering class.

Outline

(1) Introduce the category $\mathrm{Mod}(\mathfrak{R})$ of modules over a ringoid $\mathfrak{R}$.

(2) Encode the categories ${}_R\mathrm{Mod}$ and $\mathrm{Ch}({}_R\mathrm{Mod})$ as modules over certain ringoids.

(3) Show that $\mathcal{P}_{\leq n}(\mathrm{Mod}(\mathfrak{R}))$ is a special precovering class in $\mathrm{Mod}(\mathfrak{R})$.

(4) Applications: *n*-Projective modules and some cotorsion pairs in chain complexes.
Model structures on $\mathrm{Ch}({}_R\mathrm{Mod})$.

Some references

This talk presents a part of the results appearing in:

(1) [M. P.] ***Homological dimensions and Abelian model structures on chain complexes***. To appear: Rocky Mountain Journal of Mathematics.

and inspired in the following works:

(2) [S. Tempest Aldrich, Edgar E. Enochs, Overtoun M. G. Jenda, Luis Oyonarte]. ***Envelopes and Covers by Modules of Finite Injective and Projective Dimensions***. Journal of Algebra. Vol. 242, pp. 447-459. (2001).

(3) [James Gillespie]. ***Cotorsion pairs and degreewise homological model structures***. Homology, homotopy and applications. Vol. 10, No. 1, pp. 283-304. (2008).

(4) [Mark Hovey]. ***Cotorsion pairs, model category structures and representation theory***. Mathematical Zeitschrift. Vol. 241, pp. 553-592. (2002).

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What are ringoids and modules over them?

Definition (ringoids)

A **ringoid** $\mathfrak{R}$ is is a small preadditive category.

Definition (modules over a ringoid)

Let Ab denote the category of Abelian groups. A **(left)** $\mathfrak{R}$**-module** is given by a covariant functor $M\colon \mathfrak{R} \longrightarrow \mathrm{Ab}$.

Definition (morphisms between ringoids)

Given two modules over $\mathfrak{R}$, say $M, N\colon \mathfrak{R} \longrightarrow \mathrm{Ab}$. A **morphism** from M to N is a natural transformation $M \Rightarrow N$.

Let $\mathrm{Mod}(\mathfrak{R})$ denote the Abelian category of left modules over $\mathfrak{R}$.

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Some examples

Example (Modules over a ring)

Let R be a ring.

- The category $\mathfrak{M}$ given by the collections:

$$\mathrm{Ob}(\mathfrak{M}) := \{*\}, \text{ and}$$

$$\mathrm{Hom}_{\mathfrak{M}}(*, *) := R,$$

  define a ringoid.

- In this case, we have $\mathrm{Mod}(\mathfrak{M}) = {}_R\mathrm{Mod}$.

Some examples

Example (Chain complexes over a ring)

Consider the infinite graph

$$\begin{tikzcd}
\cdots \arrow[r] & 2 \arrow[loop above, "e_2"] \arrow[r, "\partial_2"] & 1 \arrow[loop above, "e_1"] \arrow[r, "\partial_1"] & 0 \arrow[loop above, "e_0"] \arrow[r, "\partial_0"] & -1 \arrow[loop above, "e_{-1}"] \arrow[r, "\partial_{-1}"] & -2 \arrow[loop above, "e_{-2}"] \arrow[r] & \cdots
\end{tikzcd}$$

together with the following relations for every $n \in \mathbb{Z}$:

$$\begin{aligned} \partial_n \circ \partial_{n+1} &= 0, \\ \partial_n \circ e_n &= \partial_n, \\ e_{n-1} \circ \partial_n &= \partial_n, \\ e_n \circ e_n &= e_n. \end{aligned}$$

Some examples

Example (Chain complexes over a ring)

Let R be a ring. Set $\mathfrak{C}$ as the ringoid defined by the following collections of objects and morphisms:

$$\operatorname{Ob}(\mathfrak{C}) = \mathbb{Z},$$

$$\operatorname{Hom}_{\mathfrak{C}}(n,m) = \begin{cases} \langle e_n \rangle := \{k \cdot e_n : k \in \mathbb{Z}\} & \text{if } n = m, \\ \langle \partial_n \rangle := \{k \cdot \partial_n : k \in \mathbb{Z}\} & \text{if } m = n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $\operatorname{Mod}(\mathfrak{C}) = \operatorname{Ch}(\operatorname{Ab})$.

Some examples

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In this case, $\mathrm{Mod}(\mathfrak{C}) = \mathrm{Ch}(\mathrm{Ab})$. And

$$\mathrm{Ch}({}_R\mathrm{Mod}) = \mathrm{Func}(\mathfrak{C}, {}_R\,\mathrm{Mod}) = \mathrm{Func}(\mathfrak{C}, \mathrm{Func}(\mathfrak{M}, \mathrm{Ab}))$$
$$\cong \mathrm{Func}(\mathfrak{C} \otimes \mathfrak{M}, \mathrm{Ab}) = \mathrm{Mod}(\mathfrak{C} \otimes \mathfrak{M}).$$

More examples: free modules in Mod($\mathfrak{R}$)

Example (A very easy but important example)

Let $\mathfrak{R}$ be a ringoid. For every $a \in \mathrm{Ob}(\mathfrak{R})$, $\mathrm{Hom}_{\mathfrak{R}}(a, -) \colon \mathfrak{R} \longrightarrow \mathrm{Ab}$ is a left module over $\mathfrak{R}$.

More examples: free modules in $\mathrm{Mod}(\mathfrak{R})$

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Definition (Free modules over a ringoid)

An $\mathfrak{R}$-module F is **free** if it is isomorphic (as objects in $\mathrm{Mod}(\mathfrak{R})$) to a coproduct of $\mathfrak{R}$-modules $\mathrm{Hom}_{\mathfrak{R}}(a_i, -)$ for a family $\{a_i\}_{i \in I}$ of objects of $\mathfrak{R}$:

$$M \cong \bigoplus_{i \in I} \mathrm{Hom}_{\mathfrak{R}}(a_i, -).$$

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Zig-zag procedure in $\text{Mod}(\mathfrak{R})$

Recall:

Main Theorem

The class $\mathcal{P}_{\leq n}(\text{Mod}(\mathfrak{R}))$ *of n-projective modules over a ringoid* $\mathfrak{R}$ *is special precovering.*

Zig-zag procedure in Mod($\mathfrak{R}$)

Recall:

Main Theorem

The class $\mathcal{P}_{\leq n}(\mathrm{Mod}(\mathfrak{R}))$ *of n-projective modules over a ringoid* $\mathfrak{R}$ *is special precovering.*

For this, we need:

Main Lemma

Let M be a n-projective $\mathfrak{R}$-module. Then for every “homogeneous” element $x \in M(a)$ there exists a “small submodule” $N \hookrightarrow M$ such that:

(1) $x \in N(a)$.

(2) *The $\mathfrak{R}$-modules N and M/N are n-projective.*

What do we mean by “elements” of an $\mathfrak{R}$-module?

Definition (Homogeneous elements of a module)

If M is an $\mathfrak{R}$-module, we say that x is a **homogeneous element of** M **of grade** a if $x \in M(a)$, with $a \in \mathrm{Ob}(\mathfrak{R})$. This will be denoted by $|x| = a$.

What does it mean that N is a "submodule" of M?

For every $\mathfrak{R}$-module $M\colon \mathfrak{R} \longrightarrow \mathrm{Ab}$ and for every pair $a, b \in \mathrm{Ob}(\mathfrak{R})$, there is an induced multiplication

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{R}}(a,b) \times M(a) &\to M(b)\\ (r,x) &\mapsto r \cdot x := M(r)(x).\end{aligned}$$

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Definition (Submodules over a ringoid)

If M is a left $\mathfrak{R}$-module, a family $N = \{N(a) : a \in \mathrm{Ob}(\mathfrak{R})\}$ of subgroups $N(a) \subseteq M(a)$ is a **submodule of** M if

$$x \in N(a) \Longrightarrow r \cdot x \in N(b), \text{ for every } r \in \mathrm{Hom}_{\mathfrak{R}}(a,b).$$

And what does it mean that an $\mathfrak{R}$-module is “small”?

Definition (Small $\mathfrak{R}$-modules)

Let κ be an infinite cardinal such that:

$$\kappa > \mathrm{Card}(\mathrm{Hom}_{\mathfrak{R}}(a, b)), \ \forall a, b \in \mathrm{Ob}(\mathfrak{R}).$$

An $\mathfrak{R}$-module M is **κ-small** if $\mathrm{Card}(M(a)) \leq \kappa$, for every $a \in \mathrm{Ob}(\mathfrak{R})$.

Characterization of free modules

Definition (admissible elements)

A linear combination of homogenous elements $y = \sum_{i \in I} r_i \cdot x_i$ is **admissible** if y is homogenous and $r_i \in \mathrm{Hom}_{\mathfrak{R}}(|x_i|, |y|)$ for every $i \in I$.

Characterization of free modules

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Definition (basis)

A family $\{x_i\}_{i \in I}$ of homogenous elements of an $\mathfrak{R}$-module M is a **basis** of M if every homogenous element $x \in M$ can be written uniquely as an admissible linear combination $x = \sum_{i \in I} r_i \cdot x_i$.

Characterization of free modules

An $\mathfrak{R}$-module is free iff
it admits a basis

$\Downarrow$

Main Lemma $\Rightarrow$ Every n-projective $\mathfrak{R}$-module is **filtered** by the set of small n-projective $\mathfrak{R}$-modules

$\Downarrow$

Main Theorem $\Leftarrow$ $\mathcal{P}_{\leq n}(\mathrm{Mod}(\mathfrak{R}))$ is the left half of a cotorsion pair cogenerated by the set of small n-projective $\mathfrak{R}$-modules

Applications to model categories

Notation and facts

(1) *If $\mathcal{X}$ is a class of objects in an Abelian category $\mathcal{C}$, $\widetilde{\mathcal{X}}$ denotes the class of exact chain complexes X with $Z_m(X) \in \mathcal{X}$, for every $m \in \mathbb{Z}$.*

(2) $\mathcal{P}_{\leq n}(\mathrm{Ch}({}_R\mathrm{Mod})) = \widetilde{\mathcal{P}_{\leq n}}$.

(3) *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in $\mathcal{C}$, then*

- $\mathrm{dg}\widetilde{\mathcal{A}}$ *is the class of chain complexes X such that $X_m \in \mathcal{A}$ for every $m \in \mathbb{Z}$, and such that $X \to B$ is null homotopic for every $B \in \widetilde{\mathcal{B}}$.*
- $\mathrm{dg}\widetilde{\mathcal{B}}$ *is defined dually.*

Applications to model categories

$(\mathcal{P}_{\leq n}(\text{Mod}(\mathfrak{R}), \mathcal{P}_{\leq n}(\text{Mod}(\mathfrak{R}))^{\perp})$
is a complete cotorsion
pair in $\text{Mod}(\mathfrak{R})$

Applications to model categories

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$\mathfrak{R} = \mathfrak{M} \; \Swarrow$

[Enochs et al., 2001]
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Applications to model categories

$(\mathcal{P}_{\leq n}(\text{Mod}(\mathfrak{R}), \mathcal{P}_{\leq n}(\text{Mod}(\mathfrak{R}))^{\perp})$
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$\Searrow \mathfrak{R} = \mathfrak{C} \otimes \mathfrak{M}$

$(\widetilde{\mathcal{P}_{\leq n}}, (\widetilde{\mathcal{P}_{\leq n}})^{\perp})$ is
a complete cotorsion pair

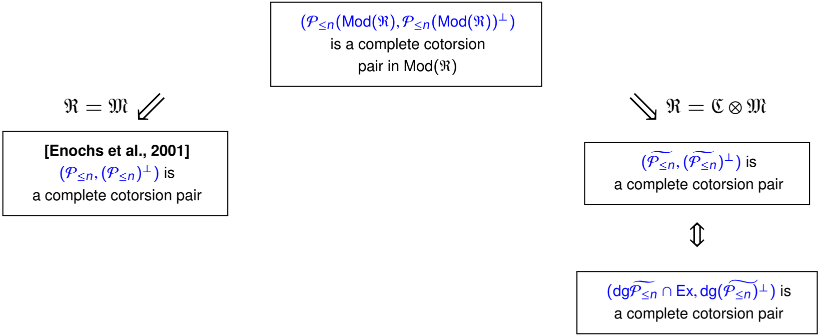
Applications to model categories

Proposition (J. Gillespie)

Suppose $(\mathcal{A}, \mathcal{B})$ *is a hereditary cotorsion pair in an Abelian category* $\mathcal{C}$. *If* $\mathcal{C}$ *has enough injective objects, then*

$$\widetilde{\mathcal{A}} = \mathrm{dg}\widetilde{\mathcal{A}} \cap \mathrm{Ex}.$$

Applications to model categories



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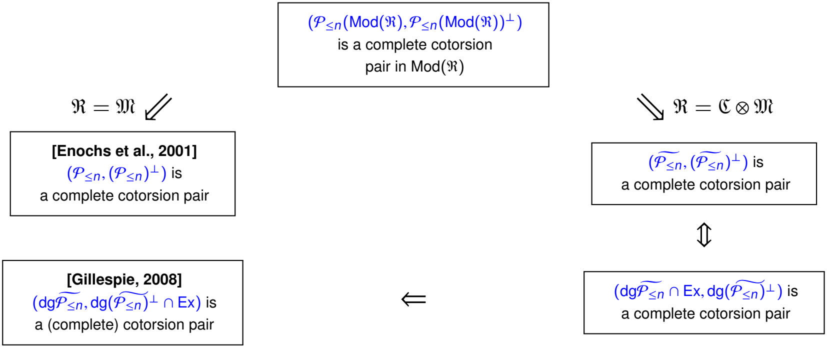
Applications to model categories



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$\mathfrak{R} = \mathfrak{M}$ $\swarrow$

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$(\mathcal{P}_{\leq n}, (\mathcal{P}_{\leq n})^{\perp})$ is
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$\Updownarrow$

[Gillespie, 2008]
$(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$ is
a (complete) cotorsion pair

$\Leftarrow$

$(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}} \cap \mathrm{Ex}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp})$ is
a complete cotorsion pair

Applications to model categories

Theorem (Hovey's correspondence - But just one part of it)

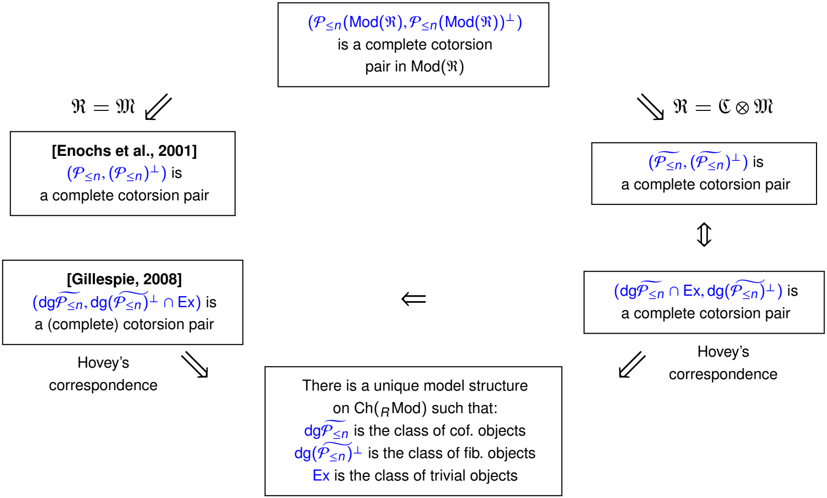
Let $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{W}$ be three classes of objects in a bicomplete Abelian category $\mathcal{C}$ such that:

- $\mathcal{W}$ *is* ***thick****.*
- $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ *and* $(\mathcal{Q}, \mathcal{R} \cap \mathcal{W})$ *are complete cotorsion pair.*

Then there exists a unique Abelian model structure on $\mathcal{C}$ such that:

- $\mathcal{Q}$ *is the class of cofibrant objects.*
- $\mathcal{R}$ *is the class of fibrant objects.*
- $\mathcal{W}$ *is the class of trivial objects.*

Applications to model categories



$(\mathcal{P}_{\leq n}(\mathrm{Mod}(\mathfrak{R}), \mathcal{P}_{\leq n}(\mathrm{Mod}(\mathfrak{R}))^{\perp})$
is a complete cotorsion
pair in $\mathrm{Mod}(\mathfrak{R})$

$\mathfrak{R} = \mathfrak{M}$ $\swarrow$

$\searrow$ $\mathfrak{R} = \mathfrak{C} \otimes \mathfrak{M}$

[Enochs et al., 2001]
$(\mathcal{P}_{\leq n}, (\mathcal{P}_{\leq n})^{\perp})$ is
a complete cotorsion pair

$(\widetilde{\mathcal{P}_{\leq n}}, (\widetilde{\mathcal{P}_{\leq n}})^{\perp})$ is
a complete cotorsion pair

$\Updownarrow$

$(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}} \cap \mathrm{Ex}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp})$ is
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[Gillespie, 2008]
$(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$ is
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$\Leftarrow$

Hovey's
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$\swarrow$ Hovey's
correspondence

There is a unique model structure
on $\mathrm{Ch}({}_R\mathrm{Mod})$ such that:
$\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}$ is the class of cof. objects
$\mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp}$ is the class of fib. objects
Ex is the class of trivial objects

Completeness of $(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$

Proposition (Compatibility and completeness)

Let $\mathcal{C}$ be an Abelian category with enough projective and injective objects, and let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{W}$ be three classes of objects in $\mathcal{C}$ such that $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ and $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ are cotorsion pairs.
If $({}^{\perp}\mathcal{W}, \mathcal{W})$ and $(\mathcal{W}, \mathcal{W}^{\perp})$ are complete cotorsion pairs in $\mathcal{C}$, then $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ is complete if, and only if, $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ is.

Completeness of $(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$

Example (of $\mathcal{W}$)

- **García Rozas:** The class Ex of exact chain complexes: (DG-projectives, Ex) and (Ex, DG-injectives) are complete cotorsion pairs in $\mathrm{Ch}({}_R\mathrm{Mod})$.

Completeness of $(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$

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- **García Rozas:** The class Ex of exact chain complexes: (DG-projectives, Ex) and (Ex, DG-injectives) are complete cotorsion pairs in $\mathrm{Ch}({}_R\mathrm{Mod})$.
- **Enochs & Jenda:** The class $\mathcal{U} = \{U \in \mathrm{Ob}({}_R\mathrm{Mod}) : \mathrm{id}(U) < \infty\}$: (Gorenstein-projectives, $\mathcal{U}$) and ($\mathcal{U}$, Gorenstein-injectives) are complete cotorsion pairs in ${}_R\mathrm{Mod}$, provided R is a Gorenstein ring.

Completeness of $(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$

Example (of $\mathcal{W}$)

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- **Enochs & Jenda:** The class $\mathcal{U} = \{U \in \mathrm{Ob}({}_R\mathrm{Mod}) : \mathrm{id}(U) < \infty\}$: (Gorenstein-projectives, $\mathcal{U}$) and ($\mathcal{U}$, Gorenstein-injectives) are complete cotorsion pairs in ${}_R\mathrm{Mod}$, provided R is a Gorenstein ring.
- **Gillespie:** The class $\mathcal{V} = \{V \in \mathrm{Ob}({}_R\mathrm{Mod}) : \mathrm{FP\text{–}id}(V) < \infty\}$: (Ding-projectives, $\mathcal{V}$) and ($\mathcal{V}$, Ding-injectives) are complete cotorsion pair in ${}_R\mathrm{Mod}$, provided R is a Ding-Chen ring.

Completeness of $(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$

Knowing that $(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$ and $(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}} \cap \mathrm{Ex}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp})$ are cotorsion pairs, where the latter is complete, we have:

Corollary

$(\mathrm{dg}\widetilde{\mathcal{P}_{\leq n}}, \mathrm{dg}(\widetilde{\mathcal{P}_{\leq n}})^{\perp} \cap \mathrm{Ex})$ *is a complete cotorsion pair.*

¡Gracias por su atención!

These slides, plus thesis and papers (containing details omitted here) available from:

http://web.mit.edu/maperez/www/