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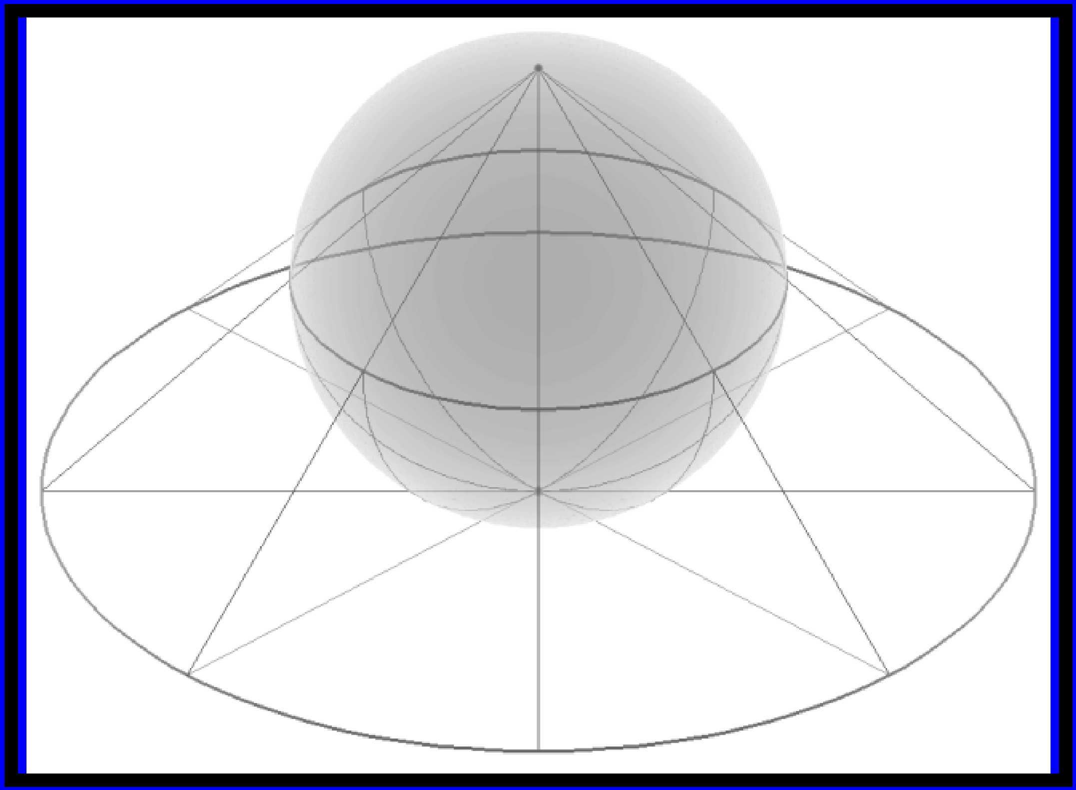
COMPLEX GEOMETRY

Course notes



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These notes are based on a course given by Steven Lu in Fall 2011 at UQÀM. All errors are responsibility of the author.

On the cover: a picture of the Riemann Sphere

(taken from: *http://en.wikipedia.org/wiki/Riemann_sphere*).

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Chapter 1

COMPLEX ANALYSIS

1.1 Complex Analysis in one variable

Let $U \subseteq \mathbb{C} = \mathbb{R}^2$ be an open subset of the complex plane. We shall denote an element $z \in \mathbb{C}$ by $z = x + iy$, where $i = \sqrt{-1}$. An function $f : U \longrightarrow \mathbb{C}$ is **holomorphic** on U if it is complex differentiable at all points of $\mathcal{U}$, i.e.,

$$\boxed{f'(z_0) = \frac{df}{dz}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

exists for every $z_0 \in \mathcal{U}$. We shall denote this by $f \in \mathcal{O}(U)$. If $S \subseteq \mathbb{C}$ is any subset, we shall say that f is holomorphic on S ($f \in \mathcal{O}(S)$) if f is holomorphic on a open neighbourhood of S.

If the function f is $\mathbb{R}$-differentiable on U then $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}$ makes sense and $df(x, y) \in \mathrm{Hom}_{\mathbb{R}}(T_{z=x+iy}U, \mathbb{R}^2)$. Recall that

$$dz = dx + idy \quad \text{and} \quad d\overline{z} = dx - idy$$

Using these expressions, we can write the differential df as

$$\boxed{df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\overline{z} = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}}$$

Notice the following relations

$$\boxed{\overline{\left(\frac{\partial f}{\partial z}\right)} = \frac{\partial \overline{f}}{\partial \overline{z}} \quad \text{and} \quad \overline{\left(\frac{\partial f}{\partial \overline{z}}\right)} = \frac{\partial \overline{f}}{\partial z}}$$

Recall that

$$\begin{gathered} f \text{ is complex differentiable} \\ \Longleftrightarrow \\ f \text{ is } \mathbb{R}\text{-differentiable and } \frac{\partial f}{\partial \overline{z}} = 0 \text{ (Cauchy-Riemann condition)} \\ \Longleftrightarrow \\ \frac{\partial u}{\partial \overline{z}} = -i\frac{\partial v}{\partial \overline{z}} \\ \Longleftrightarrow \\ df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \text{ is a rotation matrix up to a real scalar multiple } (u_x = v_y \text{ and } u_y = -v_x). \end{gathered}$$

We shall denote $V \subset\subset U$ if $\overline{V} \subseteq U$ is compact (V is **precompact** in U) and ∂V is rectifiable, i.e., ∂V is piecewise smooth.

Theorem 1.1.1 (Cauchy)**.** $f \in \mathcal{O}(U)$ if and only if $\int_{\partial V} f = 0$, for every $V \subset\subset U$ simple connected.

Theorem 1.1.2 (Cauchy's Integral Formula)**.** $z_0 \in V \subset\subset U$ if and only if

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial V} \frac{f(z)}{z - z_0} dz.$$

If $V = D_\epsilon(z_0)$ is a disk centered at z_0 of radius ϵ, then we shall denote the previous integral by

$$f(z_0) = \text{Avg}_{\partial V}(f) := \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \epsilon e^{i\theta}) d\theta.$$

Corollary 1.1.1 (Liouville Theorem)**.** Every holomorphic function on $\mathbb{C}$ is constant.

Proof: Let $V_\epsilon = B_\epsilon(z_0)$, $z_0 \in \mathbb{C}$. We show that $f'(z_0) = 0$. Using the Cauchy's Integral formula, we have

$$f'(z_0) = \frac{1}{2\pi i} \int_{\partial V_\epsilon} \frac{f(z)}{(z - z_0)^2}$$

Notice that $|f|$ is bounded on ∂V_ϵ. Then $|f(z)| \leq M$ on ∂V_ϵ for some $M > 0$. So we have

$$\begin{aligned} |f'(z_0)| &= \frac{1}{2\pi} \left| \int_{\partial V_\epsilon} \frac{f(z)}{(z - z_0)^2} dz \right| \leq \frac{1}{2\pi\epsilon^2} \int_{\partial V_\epsilon} \frac{|f(z)|}{|z - z_0|^2} dz \\ &= \frac{1}{2\pi\epsilon^2} \int_{\partial V_\epsilon} |f(z)| dz \leq \frac{M}{2\pi\epsilon^2} \int_{\partial V_\epsilon} dz \\ &= \frac{M}{2\pi\epsilon^2} \cdot 2\pi\epsilon = \frac{M}{\epsilon} \end{aligned}$$

It follows $|f'(z_0)| \longrightarrow 0$ as $\epsilon \longrightarrow \infty$. Hence $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$ and f is constant in $\mathbb{C}$.

□

Corollary 1.1.2 (Riemann Extension Theorem)**.** If $f \in \mathcal{O}(U - z_0)$, bounded near z_0 and continuous at z_0, then $f \in \mathcal{O}(U)$.

Proof: If $f \in \mathcal{O}(U - z_0)$ then $f \in \mathcal{O}(U - B_\epsilon(z_0))$, for some $\epsilon > 0$. Then the result follows since the Cauchy's Integral Formula still holds in this case.

□

Theorem 1.1.3 (Local Structure of $f \in \mathcal{O}(U)$)**.** If $f \in \mathcal{O}(U)$ is non-constant at $z_0 \in U$. Let

$$m = \min\{n > 0 \ / \ f^{(n)}(z_0) \neq 0\}.$$

Then there exists a bi-holomorphic function $\varphi : V \longrightarrow W$ from a neighbourhood of V of z_0 to a neighbourhood W of $0 \in \mathcal{C}$ with $\varphi(z_0) = 0$ such that

$$f(z) - f(z_0) = \varphi(z)^m, \quad \text{for every } z \in V.$$

Proof: Note that $f(z) - f(z_0) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$ and $g \in \mathcal{O}(U)$. Since the quotient $\frac{f(z)-f(z_0)}{z-z_0}$ is bounded on $U - z_0$ and continuous at z_0, we have by the Riemann Extension Theorem that $(z - z_0)^{m-1} g(z) = \frac{f(z)-f(z_0)}{z-z_0}$ is holomorphic on U. Proceeding this way, we have that $g(z) \in \mathcal{O}(U)$. We study several cases: If $n = 1$ then $f'(z_0) \neq 0$ and by the Inverse Function Theorem we can choose $\varphi(z) = f(z) - f(z_0)$. Now assume $n \neq 1$. Since $g(z_0) \neq 0$ then $g(z) \neq 0$ on a neighbourhood of z_0. So we can write $g = h^m$ on a neighbourhood V. We have

$$f(z) - f(z_0) = [h(z)(z - z_0)]^m$$

with $\varphi'(z_0) = h(z_0) \neq 0$. Hence, up to a local change of coordinates, f is locally of the form $z \mapsto z^m$ for some m. Such a number m is called the **ramification degree** of f at z_0.

□

Corollary 1.1.3 (Open Mapping Theorem)**.** If $f \in \mathcal{O}(U)$ is non-constant and U is connected, then f is an open mapping.

Corollary 1.1.4. If $f \in \mathcal{O}(U)$ and $|f|$ has a local maximum at $z_0 \in U$, where U is an open connected set, then f is constant on U.

Proof: Suppose f is not constant. Then by the Open Mapping Theorem, we have that $B_\epsilon(f(z_0)) \subseteq f(U)$ for some $\epsilon > 0$. In this neighbourhood there are some points of modulus greater that 0. Hence $f(z_0)$ is not a local maximum.

□

1.2 Analyticity

A function $f : U \longrightarrow \mathbb{C}$ is said to be **real analytic** on U if for every $z_0 = (x_0, y_0) \in U$ there exists a neighbourhood V of z_0 such that

$$\boxed{f(z) = \sum_{\alpha,\beta=0}^{\infty} a_{\alpha,\beta}(x-x_0)^{\alpha}(y-y_0)^{\beta} \quad \text{for every } z \in V.}$$

Similarly, f is said to be **complex analytic** on U if for every $z_0 \in U$ there exists a neighbourhood V of z_0 such that

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{for every } z \in V.}$$

In both cases the equality means **normal convergence** in U, i.e., uniform convergence on compacts in U.

Theorem 1.2.1. $f \in \mathcal{O}(U)$ if and only if f is complex analytic on U.

Proof: We know that

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(w)}{w-z}dw.$$

On the other hand,

$$\frac{1}{w-z} = \frac{1}{w-z}\left(\frac{1}{1-\frac{z-z_0}{w-w_0}}\right) = \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n.$$

It follows that

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} f(z)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}dw = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where

$$a_n = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw = \frac{1}{n!}f^{(n)}(z_0)$$

and $|w-z_0| = r$ on $\partial D_r(z_0)$.

□

Theorem 1.2.2. If $f \in \mathcal{O}(U)$ is non-constant, where U is connected, then $f^{-1}(0)$ is discrete in U.

Proof: Suppose $f^{-1}(0)$ is not discrete. Let γ be an isolated point in $f^{-1}(0)$ and consider the Taylor expansion of f about γ,

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}}{n!}(z-\gamma)^n$$

for every $z \in D_r(\gamma)$, where r is the radius of convergence of the series. Since γ is not isolated in $f^{-1}(0)$, there exists $z \in f^{-1}(0) \cap D_r(\gamma)$. We have

$$0 = \sum_{n=0}^{\infty}\frac{f^{(n)}}{n!}(z-\gamma)^n$$

and so $f^{(n)}(\gamma) = 0$ for every $n \geq 0$. It follows that f is constant on an neighbourhood of γ, getting a contradiction.

□

Corollary 1.2.1 (Analytic Continuation or Identity Theorem)**.** If $f = g$ on a non-discrete subset of U and $f, g \in \mathcal{O}(U)$, then $f \equiv g$ on U.

Lemma 1.2.1 (Schwartz)**.** If $f \in \mathcal{O}(\overline{\mathbb{D}})$ and $f \leq M$ on $\partial\mathbb{D}$, $|f(z)| \leq M|z|$ on $\partial\mathbb{D}_r$ for every $r \in (-\epsilon, 1)$, then $f(z) = Nz$; where $|N| < M$.

Another version: If $|f(z)| \leq M$ on $\overline{\mathbb{D}}$ and $f(0) = 0$, then $f(z) = Nz$ with $|N| < M$. Here, $\mathbb{D}$ denotes the Poncaré disk, i.e., the disk $\{z \in \mathbb{C} : |z|^2 < 1\}$ endowed with the metric

$$\delta(z, w) = 2\frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}.$$

1.3 Complex Analysis in several variables

Let $U \subseteq \mathbb{C}^n$ be an open subset.

Definition 1.3.1. A function $f : U \longrightarrow \mathbb{C}$ in $C'_{\mathbb{R}}(U)$ ($\mathbb{R}$-differentiable on U) is called **holomorphic**, denoted $f \in \mathcal{O}(U)$, if for every $u \in U$, the differential $df_u \in \mathrm{Hom}(T_uU, \mathbb{C})$ is a $\mathbb{C}$-linear map.

We prove as before the following result:

Theorem 1.3.1. The following conditions are equivalent:

(1) $f \in \mathcal{O}(U)$.

(2) For every $z_0 \in U$, f has the form

$$f(z_0 + \delta) = \sum_I a_I \delta^I \text{ (normal convergence)}$$

where $I = (i_1, \ldots, i_n)$ and $z^I = z_1^{i_1} \cdots z_n^{i_n}$.

(3) If $D = \{(\zeta_1, \ldots, \zeta_n) \;/\; |\zeta_i - a_i| < \alpha_i \in \mathbb{R}_{>0}\}$ is a poly-disk in U, then for every $z = (z_1, \ldots, z_n) \in \mathrm{Int}(D)$

$$\left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_i - a_i| = \alpha_i} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}$$

where $\delta D = \{|\zeta_i - a_i|\} = \alpha_i \subset \partial D$.

Note that if $f' \in C'_{\mathbb{R}}(U)$ then

$$df = \sum \frac{\partial f}{\partial x_i} dx_i + \sum \frac{\partial f}{\partial y_i} dy_i = \frac{1}{2} \sum \left(\frac{\partial f}{\partial x_i} - i \frac{\partial f}{\partial y_i} \right) dz_i + \frac{1}{2} \sum \left(\frac{\partial f}{\partial x_i} + i \frac{\partial f}{\partial y_i} \right) d\overline{z_i}$$

Theorem 1.3.2. Let $f \in \mathbb{C}^{n+1}$ such that $f(0) = 0$. Write $\mathbb{C}^{n+1} = \{(w, z_1, \ldots, z_n) = (w, z)\}$. If $f \not\equiv 0$ on the $w - axis$ $(z = 0)$ then on some neighbourhood V of 0, we have

$$f = (w, z)(w^d + a_1(z) w^{d-1} + \cdots + a_d(z))$$

where g is never zero on V.

Denote $p(w, z) = w^d + a_1(z) w^{d-1} + \cdots + a_d(z)$. Hence locally we have $\mathrm{zero}(f) = \mathrm{zero}(p)$. It follows that the roots of p are single valued holomorphic functions $w = b_i(z)$ away from the discriminant locus of f ($\Delta_f(z) = 0$, where $\Delta_f(z)$ is a polynomial in the a_i's). Hence $\{f = 0\}$ is a étale cover that covers the hyperplane $\{w = 0\} = \{(z_1, \ldots, z_n)\}$. So by induction we see that:

Fact 1.3.1. The zero of a holomorphic function is the disjoint union of submanifolds of lower dimension.

Definition 1.3.2. An **analytic set** is the set of common zeros of finitely many analytic functions.

Theorem 1.3.3 (Riemann Extension Theorem)**.**

- **Part I:** If f is a holomorphic function and bounded outside an analytic subset of codimension 2 or higher, then f extends over the subset as a holomorphic function.
- **Part II (also known as the Hartogs Theorem):**
  (1) Let $U = \Delta(r) = \{(z_1, z_2) \ / \ |z_1| < r \text{ and } |z_2| < r\}$ and $V = \Delta(r')$, such that $r' < r$ and $V \subset\subset U$, then every $f \in \mathcal{O}(U - V)$ extends to a holomorphic function on U.
  (2) If $S \subseteq U \subseteq \mathbb{C}^n$ has complex codimension greater or equal than 2, where S is an analytic subset and $f \in \mathcal{O}(U - S)$, then f extends to a holomorphic function on U.

Proof: We only proof the second part.

(1) Take a slice $z_1 = \text{const}$. Then $U - V = \{r' < |z_2| < r\}$ on this slice. Set

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|w_2|=r} \frac{f(z_1, w_2)}{w_2 - z_2} dw_2$$

Hence $F : U \longrightarrow \mathbb{C}$ is holomorphic in z_1 since

$$\frac{\partial f}{\partial \bar{z}_1} = 0 \Longrightarrow \frac{\partial F}{\partial z_1}$$

and clearly also in z_2 (Cauchy's Integral Formula). Moreover, $F = f$ on $U - V$ by the Cauchy's Integral Formula.

(2) The a 2-dimensional slice and apply (1).

□

Theorem 1.3.4 (Open Mapping Theorem)**.** If $f \in \mathcal{O}(U)$ then f is open.

Theorem 1.3.5 (Maximum Principle)**.** $|f|$ has no local maximum unless it is locally constant there.

Theorem 1.3.6 (Analytic continuation)**.** $f = 0$ in an open subset of U and $f \in \mathcal{O}(U)$, where U is connected (or arcwise connected), then $f \equiv 0$ on U.

Proof: For every path α, the set $I = \{c \ / \ f \circ \alpha(t) = 0 \ \forall t < c\}$ is open and closed.

□

Chapter 2

RIEMANN SURFACES

2.1 Complex manifolds, Lie groups and Riemann surfaces

Definition 2.1.1. A **complex manifold** M is a topological manifold whose coordinate charts are open subsets of $\mathbb{C}^n$, such that the transition maps are holomorphic. The number n is called the dimension of M. A **Riemann surface** is a complex manifold of dimension 1. A **complex Lie group** is a group that is a complex manifold such that the product and inversion maps are holomorphic.

Remark 2.1.1. Manifolds are always connected unless otherwise specified.

Example 2.1.1. The following sets are complex manifolds:

(1) $\mathbb{CP}^n = \{[z_0 : \cdots : z_n]\} = \mathbb{C}^{n+1} - \{0\}/\sim$, where

$$(z_0, \ldots, z_n) \sim (z'_0, \ldots, z'_n) \iff z = tz', \text{ for some } t \in \mathbb{C}^*.$$

Let $U_0 = \{[z_0 : \cdots : z_n] \;/\; z_0 \neq 0\}$. Let $\varphi_0 : U_0 \longrightarrow \mathbb{C}^n$ be the map given by

$$[z_0 : \cdots : z_n] \mapsto \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right)$$

which we shall call the **0-th affine chart**. Note that there exist $n+1$ affine charts that cover $\mathbb{CP}^n$.

(2) The **compact complex torus** $\mathbb{C}^n/\Gamma$, where $\Gamma \cong \mathbb{Z}^n$.

(3)

$$\begin{aligned}\mathbb{P}GL(1,0) = \mathrm{Aut}(\mathbb{CP}^1) &= \text{ set of Moebius transformations} \\ &= \left\{\frac{az_0 + bz_1}{cz_0 + dz_1} \;/\; ad - bc \neq 0\right\}/\{\pm 1\}\end{aligned}$$

The following sets are Riemann surfaces:

(4) $\mathbb{CP}^1$.

(5) $\mathbb{C}$ and $\mathbb{C}^* = \mathbb{C} - \{0\}$.

(6) The half-plane model $\mathbb{H} = \{z \ / \ \text{Im}(z) > 0\}$.

(7) The Poincaré disk model $\Delta = \mathbb{D} = \{z \ / \ |z| < 1\}$ and $\Delta^* = \Delta - \{0\}$. The models $\mathbb{H}$ and $\mathbb{D}$ are related by the map

$$z \mapsto \frac{z - a}{z - \overline{a}}$$

The exponential function $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$ is a universal covering.

Definition 2.1.2. The manifold $\mathbb{C}/\Gamma$ (and more generally its nontrivial holomorphic images) is called an **elliptic curve**. The manifold $\mathbb{CP}^1$ is called a rational curve.

Note that $\mathbb{CP}^1$ has positive curvature, $\mathbb{C}$ has zero curvature (in other words, $\mathbb{C}$ is said to be **flat**), and $\mathbb{H}$ and $\mathbb{D}$ have negative curvature. Note that Γ is a lattice $\{n + \alpha \ / \ \alpha \in \mathbb{H}\}$. The **parameter space of elliptic curves** is the quotient $\mathbb{H}/SL(2, \mathbb{Z})$. Note that $\mathbb{C}/\Gamma$ has genus 1.

Example 2.1.2.

(1) Let $f(z_0, \ldots, z_n)$ be a homogeneous polynomial. Then

$$C = V(f) = \{[z_0 : z_1 : z_2] \ / \ f(z_0, z_1, z_2) = 0\} \subseteq \mathbb{CP}$$

is called an **algebraic plane curve** over $\mathbb{C}$. This cuve C is **smooth** (or **non-singular**) if it is a submanifold (only need to check $df(p) \neq 0$ for every $p \in C$ to have C smooth).

(2) $x^d + y^d + z^d$ gives the Fermat curve of degree d in $\mathbb{CP}^2$. It is smooth since $df \neq (0, 0, 0)$ on C, $df = (x^{d-1}dx, y^{n-1}dy, z^{n-1}dz)$.

2.2 Holomorphic maps

Definition 2.2.1. Let $f : X \longrightarrow Y$ be a continuous map of topological spaces, the inverse image $f^{-1}(p)$ is called the **fibre** of f at $p \in Y$. The map f is called **discrete** if all its fibres are discrete in X.

Theorem 2.2.1 (Identity Theorem)**.** If $f_1, f_2 : S_1 \longrightarrow S_2$ are mappings of Riemann surfaces such that they coincide on a non-discrete subset of S_1, then $f_1 \equiv f_2$.

Theorem 2.2.2. Any non-constant mapping of Riemann surfaces is discrete.

The Open Mapping Theorem implies the following result:

Theorem 2.2.3. Let $f : S_1 \longrightarrow S_2$ be a non-constant map of Riemann surfaces. Assume that S_1 is compact. Then f is surjective and S_2 is compact. Furthermore, if f is **proper** (i.e., $f^{-1}(C)$ is compact for every compact set C) and discrete with finite fibres (i.e., a **finite map**) then f is called a **branched covering map** (at a branch f looks like $z \mapsto z^d$, where d is called the **branched degree**).

Let $p \in S$ be a ramification point. At such a point we call the **multiplicity** (or the **ramification degree**) of f

$$\boxed{\mathrm{mult}_p(f) = d.}$$

The degree of f is defined by

$$\boxed{\deg(f) = \sum_{p \in F} \mathrm{mult}_p(f)}$$

for every fibre F. The **ramification index** of f at p is

$$\boxed{r_p(f) = \mathrm{mult}_p(f) - 1.}$$

A map is said to be **unramified** if $r_p(f) = 0$ for every $p \in S$.

2.3 Meromorphic functions and differentials

Definition 2.3.1. A **meromorphic function** on a Riemann surface Z is a holomorphic function on an open subset $U \subseteq Z$ where $Z - U$ is a discrete set consisting of at most poles of the function.

Recall that a **pole** $p \in Z - U$ is defined by one of the following equivalent conditions:

(a) $\lim_{z \to p} f(z) = \infty$.

(b) f can be written locally as a Laurent series

$$f(z) = \sum_{-\infty}^{\infty} a_i z^i$$

with $a_i = 0$ for every $i < n \in \mathbb{Z}$.

(c) $f = g/h$, where $g, h \in \mathcal{O}(p)$, $g(p) \neq 0$ and $h(p) = 0$.

The set of such functions is denoted $\mathcal{M}(Z)$. We have

$$f \in \mathcal{M}(Z) \Longleftrightarrow f : Z \xrightarrow{\text{hol}} \mathbb{CP}^1$$

and that

$$\text{poles of } f = f^{-1}(\infty)$$

Example 2.3.1.

(1) A non-constant polynomial defines a meromorphic function from $\mathcal{CP}^1$ with pole order at ∞ equal to $\deg(f) \geq 1$.

(2) A rational function $p(z)/g(z)$ defines a meromorphic function with pole order at ∞ equal to $\deg(p) - \deg(g)$. If this difference is negative then f has a zero at ∞.

Fact 2.3.1. $\mathcal{M}(Z)$ is a field.

A finite map of Riemann surfaces $f : Z_1 \longrightarrow Z_2$ corresponds to a finite field extension

$$f^* : \mathcal{M}(Z_2) \hookrightarrow \mathcal{M}(Z_1).$$

Definition 2.3.2. A **meromorphic differential** on a Riemann surface Z is a holomorphic differential ω on an open $U \subseteq Z$ whose complement $Z - U$ is discrete and consist of poles of ω. Locally, $\omega = f dz$ even at a pole. The **pole order** of ω is defined by that of f (locally) and its residue at p is the same as that of $f dz$ $(p = 0)$, denoted $\text{Res}_p(\omega)$.

Theorem 2.3.1 (Residue)**.** $V \subset\subset Z$ with rectifiable boundary ∂V and ω differentiable on Z. Then

$$\int_{\partial V} \omega = \sum_{p \in V} \text{Res}_p(\omega)$$

We shall denote the space of meromorphic differentials by $\mathcal{M}'(Z)$.

Theorem 2.3.2. $\omega \in \mathcal{M}'(Z)$, $\sum_{p\in Z} \mathrm{Res}_p(w) = 0$.

Corollary 2.3.1 (Sum rule or product rule, Reciprocity Theorem)**.** If Z is compact and $f \in \mathcal{M}(Z)$, then

$$\#\mathrm{zero}(f) = \#\mathrm{poles}(f)$$

on Z counting multiplicity.

Corollary 2.3.2. If $f \in \mathcal{M}(\mathbb{CP}^1)$ then f is rational. Hence $\mathcal{M}(\mathbb{CP}^1) = \mathbb{C}(Z)$.

Corollary 2.3.3. $\mathcal{M}'(\mathbb{CP}') = \mathbb{C}(Z)dz$.

Let S be a Riemann surface and $U \subset\subset S$ a relative compact open subset of S with good ∂U. Let ω be a meromorphic differential ($\omega \in \mathcal{M}'(S)$). Then we have

$$\int_{\partial U} \omega = \sum_{p\in U} \mathrm{Res}_p\omega$$

If $\partial U = \emptyset$ (so $U = S$ is compact) then

$$\sum_{p\in S} \mathrm{Res}_p\omega = 0$$

We have that if S is compact then $\mathcal{S} = R(S)$, the set of rational functions (later, we are going to study this in detail).

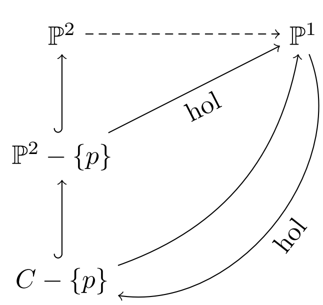
Example 2.3.2. $\mathcal{M}(\mathbb{P}^1) = R(\mathbb{P}^1)$, where $\mathbb{P}^1 = \mathbb{CP}^1$. Note that $\mathbb{CP}^1$ is compact and $\mathbb{CP}^1 = \mathbb{C} \cup \mathbb{C}$, where there is a map between charts

$$z \in \mathbb{C} \overset{w=\frac{1}{z}}{\mapsto} w \in \mathbb{C}$$

Since $\sum_{p\in S} \mathrm{Res}_p\omega = 0$, we have $\omega = \frac{df}{f}$, and $\mathrm{Res}_p\omega = \mathrm{ord}_p f$.

Recall that $f \in \mathcal{M}(\mathbb{P}^1)$ if and only if $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is holomorphic.

Example 2.3.3. Let C be a quadratic (conic) curve in $\mathbb{CP}^2 = \mathbb{P}^2$. Let $p \in C$. The space of lines passing through p is $\mathbb{P}^2$ and so gives a meromorphic map $\mathbb{P}^2 \longrightarrow \mathbb{P}^1$. We have the following diagram:



This shows that $C \cong_{\text{hol}} \mathbb{P}^1$. For example, we have the Fermat curve $z_0^2 + z_1^2 = z_2^2$. If $\mathbb{P}^1 = \{[u : v]\}$, then define a map $z_0 = (u - v)^2$, $z_1 = 2uv$ and $z_2 = (u + v)^2$.

Note that $\pi_1(S) = 0$ where $S = \mathbb{P}^1, \mathbb{C}, \mathbb{D}$. Recall that if $\pi_1(S) = \mathbb{Z}$ then S is not compact.

Example 2.3.4. $S_1 = \mathbb{C}^*$, $S_2 = \mathbb{D} - \frac{1}{2}\mathbb{D}$. These two examples are not biholomorphic Riemann surfaces. Neither $\mathbb{D}^*$ is biholomorphic to $\mathbb{C}^*$. If so, then a biholomorphic function $\mathbb{D}^* \longrightarrow \mathbb{C}^*$ produces an extension $\mathbb{D} \longrightarrow \overline{S}$, getting a contradiction.

The Riemann surfaces $\mathbb{H}$ and $\mathbb{D}$ are biholomorphic via the map

$$\omega \mapsto \frac{w - a}{w - \overline{a}}$$

2.4 Weierstrass P-function on $\mathbb{C}$

Consider the torus $\mathbb{C}/\Gamma$, where

$$\Gamma = \langle 1, \tau \rangle_{\tau \in \mathbb{C}} = \mathbb{Z} \oplus \mathbb{Z}\tau$$

is a lattice in $\mathbb{C}$. Define the **Weirstrass P-function** by the formula:

$$\boxed{\mathcal{P}(z) = p(z, \Gamma) = \frac{1}{z^2} + \sum_{\lambda \in \Gamma - \{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right]}$$

Hence $\mathcal{P} \in \mathcal{M}(\mathbb{C})$ and satisfies:

(a) $\mathcal{P}(z) = \mathcal{P}(-z)$,

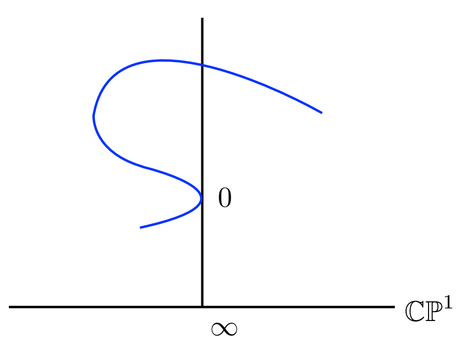
(b) $\mathcal{P}(z + \lambda) = \mathcal{P}(z)$ for every $\lambda \in \Gamma$,

(c) there exists no other poles.

Hence $\mathcal{P}$ descends to a meromorphic function on $C = \mathbb{C}/\Gamma$ with a pole at 0 (degree $= 2$), i.e., we have a holomorphic function

$$C \xrightarrow{f} \mathbb{CP}^1$$

Locally, this map looks like $z \mapsto z^n$ about 0. In this case $n = 2$. This map has exactly degree 2 since there are no other poles. Moreover, f is branched at 0.



Now $\mathcal{P}'(z)$ is also periodic (period Γ) with triple poles on Γ and no other poles. The map $C \longrightarrow \mathbb{P}^2$ given by

$$z \in \mathbb{C} \mapsto [\mathcal{P}(z) : \mathcal{P}'(z) : 1]$$

defines a holomorphic function $C - \{0\} \longrightarrow \mathbb{P}^2$,

$$[\mathcal{P}(z) : \mathcal{P}'(z) : 1] = \left[\frac{\mathcal{P}(z)}{\mathcal{P}'(z)} : 1 : \frac{1}{\mathcal{P}'(z)} \right]$$

and it extends over 0.

2.5 Dimension on Riemann surfaces

Definition 2.5.1. A **divisor** D on a Riemann surface S is a formal $\mathbb{Z}$-linear combination of points in S

$$\boxed{D = \sum a_i P_i}$$

where $a_i \neq 0$ for every i and $\{P_i\}$ is a discrete subset of S. A divisor D is called **effective** if $a_i \geq 0$ for every i.

If $\text{supp}(D) = \{P_i \ / \ a_i \neq 0\}$ is finite, then

$$\boxed{\deg(D) := \sum a_i}$$

Definition 2.5.2. Let $f \in \mathcal{M}(S) - \{0\} = \mathcal{M}^*(S)$. The **divisor** of f is defined by

$$\boxed{(f) := (f)_0 - (f)_\infty}$$

where

$$\boxed{(f)_0 = \sum (\text{ord}_P f)P \quad \text{and} \quad (f)_\infty = \sum_{P \in f^{-1}(\infty)} (\text{mult}_P f)P}$$

Note that $\text{mult}_P f = -\text{ord}_P f$, so we can rewrite the previos expresion as

$$(f) = \sum (\text{ord}_P f)P$$

Lemma 2.5.1. If S is a compact Riemann surface, then $\deg(f) = 0$ for every $f \in \mathcal{M}^*(S)$, where deg is a map $\text{Div}(S) \longrightarrow \mathbb{Z}$.

Definition 2.5.3. A divisor is called **principal** if it lies in the image of $\deg(\ ) : \mathcal{M}^* \longrightarrow \mathbb{Z}$.

Definition 2.5.4. Two divisors D_1 and D_2 are said to be **linearly equivalent**, denoted $D_1 \sim D_2$, if $D_1 - D_2$ is principal.

Example 2.5.1. $D_1 \sim D_2$ on $\mathbb{P}^1$ if and only if $\deg(D_1) = \deg(D_2)$.

Example 2.5.2. What condition we need if we want $D_1 \sim D_2$ on $C = \mathbb{C}/\Gamma$. Let $p, q \in C$ be two distinct points in C and suppose that $D = p - q = (f)$ for some $f \in \mathcal{M}^*$. Then f is a map $C \longrightarrow \mathbb{P}^1$ with $\deg(f) = 1$. We have $(f)_0 = p$ and f is bijective. Then f is a biholomorphic map, getting a contradiction.

Given $\omega \in \mathcal{M}'(S)^*$. Recall that this means $\omega = f dz$ for a local coordinate z at p and $f \in \mathcal{M}(p)$.

Definition 2.5.5. $\text{ord}_p \omega = \text{ord}_p f$. The divisor of the form

$$\boxed{(\omega) := \sum_{p \in S} (\text{ord}_p \omega)P}$$

is called a **canonical divisor** and is denoted K_S or simply K. A divisor is canonical if $D \sim (w)$.

Example 2.5.3.

- $K_{\mathbb{C}/\Gamma} = (dz) = 0 \cdot P$.
- $K_{\mathbb{P}^1} = (dz) = -2 \cdot \infty$, $z = \frac{1}{\omega}$, $dz = -\frac{d\omega}{\omega^2}$.

Theorem 2.5.1 (Riemann - Hurewicz)**.** If $f : S_1 \longrightarrow S_2$ is a finite map, then $K_{S_1} \sim K + S_2 + R$ where R is the ramification divisor

$$R = \sum_{p \in S_1} r_p(f) P,$$

where each $r_p(f) \geq 0$.

Proof: Locally, f looks like $z \mapsto z^n$, and $dz^n = nz^{n-1}dz$.

□

Corollary 2.5.1 (Riemann - Hurewicz formula)**.**

$$2 \cdot g(S_1) - 2 = \deg K_{S_1} = (\deg f) \cdot \deg K_{S_2} + \deg R,$$

where $K_{S_1} = (\omega)$, $\omega \in \Gamma(T^\vee S_1)$ (global sections of the tangent bundle) $\longleftrightarrow \omega^\vee \in \Gamma(TS_1)$.

2.6 Covering spaces

Recall that an **étalé space** (or étale map) over X is a continuous map $p : \widetilde{X} \longrightarrow X$ such that p is a local homeomorphism: that is, for every $x \in \widetilde{X}$, there is an open set U in $\widetilde{X}$ containing x such that the image $p(U)$ is open in X and the restriction of p to U is a homeomorphism $p|_U : U \longrightarrow p(U)$. A connected covering space $p : \widetilde{X} \longrightarrow X$ is a **universal** cover if $\widetilde{X}$ is simply connected. The name universal cover comes from the following important property: if the map $p : \widetilde{X} \longrightarrow X$ is a universal cover of the space X and the map $p' : X' \longrightarrow X$ is any cover of the space X where the covering space X' is connected, then there exists a covering map $f : X' \longrightarrow \widetilde{X}$ such that $p \circ f = p'$. Any manifold X has a universal cover $\widetilde{X}$ with étale covering map $f : \widetilde{X} \longrightarrow X$ and $\pi_1(X)$ acts on $\widetilde{X}$ discretely and freely with quotient f. Moreover, there exists a bijection

$$\pi_1(S) \supseteq H \mapsto \{\widetilde{X}/H \overset{\text{étale}}{\longrightarrow} X\}$$

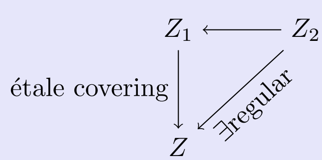
between the set of subgroups of $\pi_1(X)$ up to conjugation and the set of étale coverings from a connected manifold.

$$G/H \text{ for } H \text{ normal} \iff \text{Galois (regular) covering.}$$

Recall that a covering map $p : \widetilde{X} \longrightarrow X$ is said to be **Galois** if for every $x \in X$ and $\widetilde{x} \in p^{-1}(x)$, the subgroup $p_*\pi_1(\widetilde{X}, \widetilde{x})$ is normal in $\pi_1(X, x)$.

If X is a Riemann surface, then the complex charts on X lifts to any covering space.

Lemma 2.6.1. Let $f : Z_1 \longrightarrow Z$ be a finite étale covering corresponding to $H \hookrightarrow \pi_1(Z)$. Then there exists a finite regular covering $h : Z_2 \longrightarrow Z$ and a finite étale covering $g : Z_2 \longrightarrow Z_1$ such that $f \circ g = h$.



Proof: H has a finite number of conjugates in $\pi_1(Z)$, such a number equals $[\pi_1(Z) : N(H)]$ and these intersection is then of finite index.

□

Corollary 2.6.1. For every n, there exists a unique étale n-sheeted covering $Z \longrightarrow \mathbb{D}^*$ and it is isomorphic to $\mathbb{D}^* \longrightarrow \mathbb{D}^*$ $(z \mapsto z^n)$.

Example 2.6.1. If $f : Z_1 \longrightarrow Z_2$ is finite between Riemann surfaces, then there is a finite étale covering

$$Z_1 - f^{-1}(\Delta) \longrightarrow Z_2 - \Delta$$

where $\Delta = f(\text{supp} R_f)$ is the branching locus.

Conversely, the previous corollary implies:

Theorem 2.6.1. Let $\Delta \subseteq Z_2$ be a discrete subset. A finite étale covering $U \longrightarrow Z_2 - \Delta$, where U is an open subset, has a unique continuation to a finite map

$$U \subseteq Z_1 \longrightarrow Z_2$$

where Z_1 is a Riemann surface.

2.7 The Riemann surface of an algebraic function

Let Z_2 be a Riemann surface.

Proposition 2.7.1. Let

$$P(T) = T^n + c_1 T^{n-1} + \cdots + c_n$$

in $\mathcal{M}(Z_2)[T]$ be an irreducible polynomial. Then there exists a map of Riemann surfaces $f : Z_1 \longrightarrow Z_2$ of degree n and a meromorphic function $F \in \mathcal{M}(Z_1)$ that satisfies:

$$F^n + f^*(c_1)F^{n-1} + \cdots + f^*(c_n) = 0. \quad (*)$$

Proof: Let $\Delta \subseteq Z_2$ be the discrete set containing the poles of the c_i's and the points p where

$$P_p(T) := T^n + c_1(p)T^{n-1} + \cdots + c_n(p)$$

has multiple roots. Then $U = \{(p, z) \in (Z_2 - \Delta) \times \mathbb{C} \ / \ P_p(z) = 0\}$ is a Riemann surface, and $Z_2 - \Delta$ is a finite étale cover. We claim that it is connected, i.e.,

Claim: Given $f : Z_1 \longrightarrow Z_2$, then every $F \in \mathcal{M}(Z_1)$ is algebraic over $\mathcal{M}(Z_2)$ and satisfies an equality of the form $(*)$ but with degree less or equal than the $\deg f$.

□

Corollary 2.7.1. If $Z_1 \longrightarrow Z_2$ is finite, then

$$f^* : \mathcal{M}(Z_2) \longrightarrow \mathcal{M}(Z_1)$$

is a finite field extension.

Example 2.7.1. $\mathcal{M}(\mathbb{P}^1) = \mathbb{C}(Z)$, the field of rational functions in variable z.

Theorem 2.7.1. As soon as there exists a meromorphic function f on Z_1 ($\longleftarrow$ compact $\Longrightarrow$ f is finite), $\mathcal{M}(Z_1)$ is finite algebraic over $\mathbb{C}(z)$ of extension deg $= \deg f$.

$$Z_1 \xrightarrow{f} \mathbb{P}^1$$

Corollary 2.7.2.

(1) If $Z_1 \xrightarrow{f} Z_2$ is finite then $\mathcal{M}(Z_2) \overset{f^*}{\hookrightarrow} \mathcal{M}(Z_1)$ is a finite field extension of degree (f).

(2) Conversely, let $\mathcal{M}(Z_2) \overset{\varphi}{\hookrightarrow} \mathcal{M}(Z_1) = K$ be a finite field extension of degree d. Then there exists a finite map $Z_1 \longrightarrow Z_2$ of degree d whose field extension is isomorphic to φ.

(3) A field K of transcendental degree $= 1$ over $\mathbb{C}$ is isomorphic to $\mathcal{M}(Z)$ of some compact Riemann surface. Such a Z is called a **model** of K.

2.8 Review

Note that the ratio of two meromorphic 1-forms is a meromorphic function:

$$\omega_1, \omega_2 \in \mathcal{M}'(Z) \Longrightarrow \left(\frac{\omega_1}{\omega_2}\right) = (f) \in \mathrm{Div}_P(Z), \text{ where } f \in \mathcal{M}(Z).$$

Here, Div_P denotes the set of principal divisors. Note that $\omega \in \mathcal{M}'(Z)$ if and only if $\omega \in \Gamma_m(T^\vee Z)$, where $T^\vee Z$ denotes the cotangent bundle of Z and $\Gamma_m(T^\vee Z)$ is the set of holomorphic sections of $T^\vee Z$. Also, $T \otimes T^\vee = \mathcal{O}$ (trivial line bundle) and so

$$1 = \left(\frac{1}{\omega}\right) \cdot \omega \in \Gamma_m(\mathcal{O}) = \mathcal{O},$$

where $\frac{1}{\omega} \in TZ$ and $\omega \in T^\vee Z$.

If $\omega_1 = f\omega_2$ then $(\omega_1) = (f) + (\omega_2)$. So we get the formula

$$\deg(\ ) = \deg(f) + \deg(\ )$$

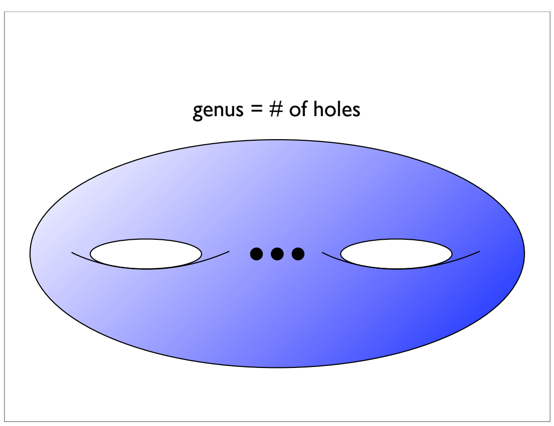
On the other hand, $\deg(f) = 0$. To show this, we know that $f(z) = z^n$ locally, where $n = \mathrm{ord}(f)$. Then $\frac{df}{f} = n\frac{dz}{z}$ and $\sum_{p \in Z} \mathrm{Res}_p \frac{df}{f} = 0$, where Z is compact. Hence we get the following result:

Theorem 2.8.1. $\deg(\omega)$ has the same value for any $\omega \in \mathcal{M}'(Z)$, assuming that Z is a compact and connected Riemann surface.

Let g be the topological genus of Z. We have the following relations:

$$\boxed{\deg(\omega) = 2g - 2 \quad \text{and} \quad \deg\left(\tfrac{1}{\omega}\right) = 2 - 2g = \chi(Z)}$$

where $\frac{1}{\omega} \in \Gamma(TZ)$.



Recall the following results:

Theorem 2.8.2. Let $P(T) = T^n + c_1 T^{n-1} + \cdots + c_n \in \mathcal{M}(Z)[T]$ be an irreducible polynomial in T over $\mathcal{Z}$. Then there exists a finite map of Riemann surfaces $f : Z' \longrightarrow Z$ of degree n, unique up to isomorphisms, and a meromorphic function F on Z' satisfying

$$F^n + (f^* c_1) F^{n-1} + \cdots + (f^* c_n) = 0$$

Corollary 2.8.1.

(1) $Z_1 \longrightarrow Z_2$ finite $\Longrightarrow f^* : \mathcal{M}(Z_2) \overset{\varphi}{\hookrightarrow} \mathcal{M}(Z_1)$ is a finite field extension of degree n.

(2) Conversely, any finite field extension of degree n gives rise to a finite map $Z_1 \longrightarrow Z_2$ whose associated field extension is isomorphic to φ.

(3) A field extension K of transcendence degree 1 (i.e., K is a finite field extension over $\mathbb{C}(Z)$) is isomorphic to $\mathcal{M}(Z)$ for some Riemann surface Z (finite over $\mathbb{CP}^1$).

Z is called a smooth model of K.

Theorem 2.8.3. Let $C \subseteq \mathbb{CP}^2$ be an irreducible plane curve, i.e., $C = V(F)$ where F is an irreducible homogeneous polynomial in $(z_0 : z_1 : z_2)$. Then there exists a compact Riemann surface Z and a generically injective map $f : Z \longrightarrow \mathbb{CP}^2$ whose image is C.

Here, generically injective means birational ($\longleftrightarrow$ isomorphic on an open set).

Proof: If F is irreducible then $\mathbb{C}[x_0, x_1, x_2]/(F, x_2 - 1)$ is an integral domain and has a quotient field K. The mapping $f(P) = (x_0(P) : x_1(P) : 1)$ extends to by continuity to a desingularization of C. □

Theorem 2.8.4 (Hurwitz)**.** If $f : Z_1 \longrightarrow Z_2$ is a finite map, then $K_{Z_1} \sim f^* K_{Z_2} + R$ (f^* = pullback) where R is the ramification divisor $\sum_{p \in Z_1} r_p(f) \cdot p$, $r_p(f) = \mathrm{ord}_p(f) - 1$, where $f : Z_1 \longrightarrow Z_2$ is locally at p of the form $f(z) = z^{\mathrm{ord}_p(f)}$ and

$$\deg(f) = \sum_{p \in \text{ fibre}} \mathrm{ord}_p(f),$$

where this sum is independent of the choice of the fibre.

Corollary 2.8.2 (Riemann-Hurwitz)**.**

$$\deg(K_{Z_1}) = (\deg(f)) \cdot (\deg(K_{Z_2})) + \deg(R)$$

Corollary 2.8.3. $g(Z_1) \geq g(Z_2)$, where g is for genus.

From this it follows that there exists no map from a rational curve to an elliptic curve.

The following equality is the algebraic geometry definition of topological genus:

$$\boxed{g(Z) = \dim(\Gamma(K_2)) = \dim_{\mathbb{C}}(\mathrm{Hol}'(Z))}$$

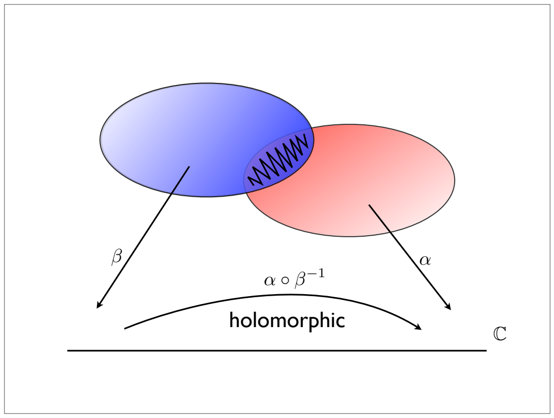
where $\mathrm{Hol}'(Z)$ is the set of holomorphic differentials on Z.

2.9 Topology of Riemann surfaces

One known fact about Riemann surfaces is that any Riemann surface is orientable. Recall that a manifold is orientable if its transition functions have positive Jacobian . Let Z be a Riemann surface, consider two intersecting charts U_α and U_β, and let $z \in U_\alpha \cap U_\beta$. Then we have

$$\frac{df}{dz}(z) = f'(z) \quad \text{as an } \mathbb{R}\text{-matrix.}$$

We write $f = u + iv$, then $df = \alpha + i\beta$.



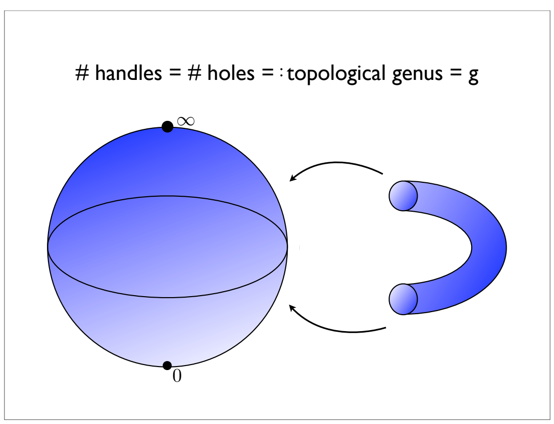
By the Cauchy-Riemann equations, we have

$$df = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

and so

$$\text{Jac}(\alpha \circ \beta^{-1}) = \det(D(\alpha \circ \beta^{-1})) = |df|^2 = \det(df) = \alpha^2 + \beta^2 > 0.$$

It is also known that every Riemann surface is obtained by attaching handles to $\mathbb{CP}^1 = S^2$.

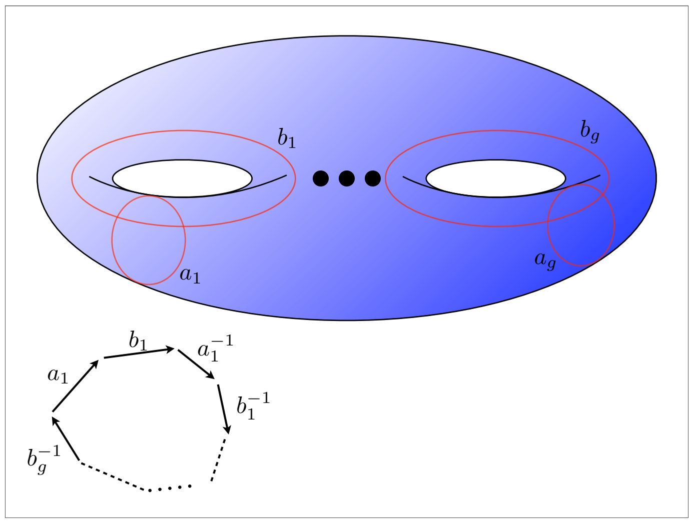


Theorem 2.9.1. Any Riemann surface is triangularizable.

This fact is easy to prove for Compact Riemann surfaces, and shows that any such is obtained by attaching handles to $\mathbb{S}^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. We denote Z_g for a Riemann surface Z of genus g. It is known that the first homotopy group of Z_g is given by

$$\boxed{\pi_1(Z_g) = \mathbb{Z}_{a_1} * \mathbb{Z}_{b_1} * \cdots * \mathbb{Z}_{a_g} * \mathbb{Z}_{b_g} / \left\langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \right\rangle}$$

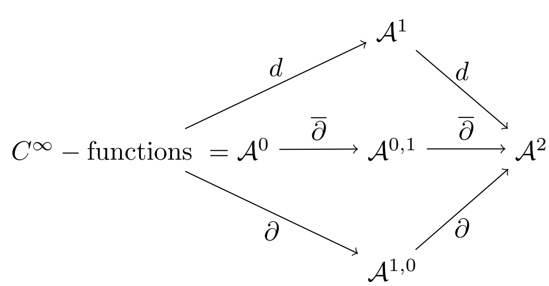
Notice that every hole gives two generators.



Let $\mathcal{A}^i$ denote the space of C^∞-complex valued i-forms on a Riemann surface, then

$$\begin{aligned} \mathcal{A} &= \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}, \\ \omega' &= f dx + g dy = h dz + h d\overline{z}, \end{aligned}$$

where $h dz \in \mathcal{A}^{1,0}$ and $h d\overline{z} \in \mathcal{A}^{0,1}$. We also have differential operators making the following diagram commutes



where $d = \partial + \overline{\partial}$.

Definition 2.9.1. An i-form $\omega \in \mathcal{A}^i$ is called **closed** if $d\omega = 0$. It is called **exact** if $\omega = d\alpha$.

Since $d \circ d = 0$, we have $\text{Im}(d) \subseteq \text{Ker}(d)$, and so we define the de Rham cohomology groups of Z and the quotient

$$\boxed{H^i_{\text{dR}}(Z) = \{\text{closed } i\text{-forms}\}/\{\text{exact } i\text{-forms}\}}$$

Note that $H^0_{\mathrm{dR}}(Z) = \mathbb{C}$ if Z is connected. If Z is also compact, by the Poincaré duality Theorem we have an isomorphism $H^2_{\mathrm{dR}}(Z) \longrightarrow H^0_{\mathrm{dR}}(Z)$ given by

$$[\omega] \mapsto \int_Z \omega$$

Also, $H^n_{\mathrm{dR}}(Z) = 0$ for every $n \geq 3$. For any compact and connected Riemann surface Z_g, the middle cohomology is given by

$$\boxed{H^1_{\mathrm{dR}}(Z_g) = \pi_1(Z_g)/\left\langle \text{commutator subgroup} \right\rangle = \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{b_1} \oplus \cdots \oplus \mathbb{Z}_{a_g} \oplus \mathbb{Z}_{b_g}}$$

Theorem 2.9.2. Let ω be a holomorphic differential ($\longleftrightarrow$ 1-form). Then ω is d-closed (hence $[\omega] \in H^1_{\mathrm{dR}}(Z)$).

Proof: Locally, $\omega = f(z)dz$ where $f \in \mathcal{O}$. Then

$$\begin{aligned} d\omega = (\partial + \overline{\partial})\omega = \partial\omega + \overline{\partial}\omega &= \left(\frac{\partial f}{\partial z} dz\right) \wedge dz + \left(\frac{\partial f}{\partial \overline{z}} d\overline{z}\right) \wedge dz \\ &= \partial f \wedge dz + \overline{\partial} f \wedge d = 0 + 0 d\overline{z} \wedge dz, \text{ since } f \text{ is holomorphic.} \end{aligned}$$

□

2.10 Product structures on $\bigoplus_i H^i_{\text{dR}}(Z)$

By taking wedge products of forms, we have a surjective map $H^1_{\text{dR}}(Z) \times H^1_{\text{dR}}(Z) \longrightarrow \mathbb{C}$ given by

$$([\omega_1], [\omega_2]) \mapsto \int_Z \omega_1 \wedge \omega_2$$

In this situation we have $H^1_{\text{dR}}(Z) \cong (H^1_{\text{dR}}(Z))^\vee$ (finite dimesnion) and so the previous map is a perfect pairing.

We also have the cap product $\cap : H_1(Z) \times H_1(Z) \longrightarrow \mathbb{Z}$ which gives rive to a perfect pairing.

By the Poincaré duality Theorem we have $H^1_{\text{dR}}(Z) \cong H_1(Z)^\vee \otimes \mathbb{C}$. Also

$$H^1_{\text{dR}}(Z) = (H_1(Z)^* = H_1(Z)) \otimes \mathbb{C}.$$

Recall that the Euler characteristic of Z is given by

$$\boxed{\chi(Z) := \sum_i (-1)^i \dim H^i_{\text{dR}}(Z) = \sum_i (-1)^i \dim_{\mathbb{Z}} H_i(Z) = 2 - 2g}$$

Recall $H^0_{\text{dR}}(Z) = H^2_{\text{dR}}(Z) = \mathbb{C}$ and $H^1_{\text{dR}}(Z) = \mathbb{C}^{2g}$.

Definition 2.10.1. Let Ω be the space of differentials (= holomorphic 1-forms) on a compact Riemann surface Z. The **geometric genus** of Z is defined by

$$\boxed{P_g(Z) = \dim_{\mathbb{C}}(\Omega)}$$

Clearly, $\Omega \hookrightarrow H^1_{\text{dR}}(Z)$ if Z is compact. We have that

$$f dz = \frac{dg}{dz} dz = dg = 0$$

implies that $g \in \mathcal{O}$. Since Z is compact, we get that $g =$ const. Similarly, $\overline{\Omega} \hookrightarrow H^1_{\text{dR}}(Z)$. It follows $2P_g \geq 2g$. It is difficult to determine when they are equal. The fact that $P_g \geq 2$ implies that there exist meromorphic functions. If $P_g = g$, then a loop C on a compact Riemann surface Z is homotopic to 0 if and only if $\int_C \omega = 0$, for every $\omega \in \Omega$. The map

$$\boxed{\omega \mapsto \int_C \omega}$$

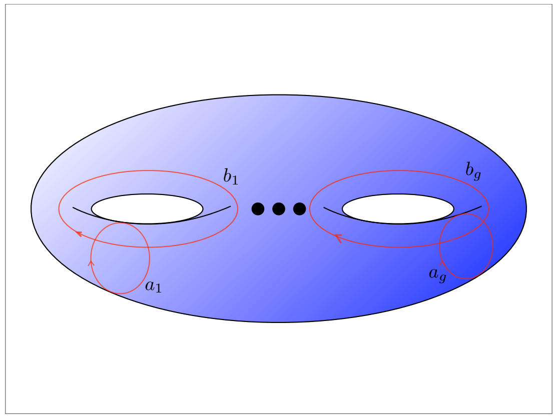
is called the **period map**.

Definition 2.10.2. Let S be a Riemann surface, define

$$H_1(S; \mathbb{Z}) = \pi_1(S)/[\pi_1(S), \pi_1(S)]$$

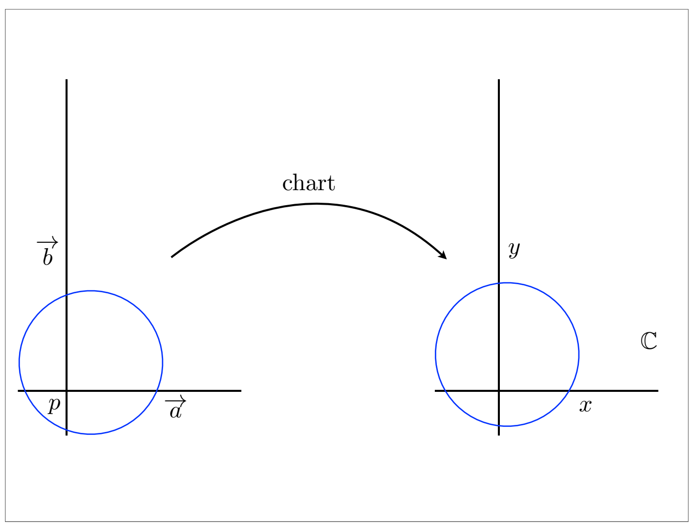
where $[\pi_1(S), \pi_1(S)]$ denotes the commutator subgroup of $\pi_1(S)$.

Note that if S is compact then $H_1(-)$ is a free $\mathbb{Z}$-module generated by a_1, b_1, $\ldots$, a_n, b_n. We can consider each a_i and b_i as maps $S^1 \longrightarrow S$ or curves with parameter $t \in [0, 1]$ in S, oriented counterclockwise.



Since S is oriented we have that there exists an intersection pairing in H_1 (any two loops, or elements in H_1, are homotopic to loops which are transversal). Note that a and b are the same element in H_1 if and only if $a + (-b) = a - b = \partial U$, for some open set $U \subseteq S$, i.e., a is homologous to b. Here $-b$ denotes reverse orientation. So if a is homotopic to b then they are homologous.

Consider the following picture:



We have $\overrightarrow{a} \wedge_p \overrightarrow{b} = kx \wedge_p y$, and denote

$$\boxed{(a,b)_p = \operatorname{sign}(k)}$$

and

$$\boxed{(a,b) = \sum_{p \in a \cap b} (a,b)_p}$$

Since the previous sum does not depend on the choice of the prepresentative, we have a well defined map

$$(\ ,\ ) : H_1 \times H_1 \longrightarrow \mathbb{Z}$$

If S is a Riemann surface of genus g, we have in terms of a basis that

$$(a_i, b_j) = (b_i, b_j) = 0, \text{ for every } i, j,$$
$$(a_i, b_j) = -(b_j, a_i) = \delta_{ij}.$$

Then we have that each (a_i, b_j) is a skew-linear pairing which is **unimodular** ($\det = 1$).

$$\begin{array}{cc} & \begin{array}{cc} a_i & b_i \end{array} \\ \begin{array}{c} a_i \\ b_i \end{array} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xrightarrow{\det} 1 \end{array}$$

Hence $H_1(S;\mathbb{Z}) = \text{Hom}(H_1(S;\mathbb{Z}),\mathbb{Z}) =: H^1(S;\mathbb{Z})$.

Let $\omega \in \mathcal{M}'(S)$ be a closed meromorphic differential. Recall that ω is exact if and only if $\omega = df$ for some meromorphic function $f \in \mathcal{M}(S)$. Can we choose a function $f \in \mathcal{M}(S)$ such that $f = \int_p^z \omega$? As an exercise, think if it is possible that $\text{Res}_p(\omega) = 0$ for every $p \in S$. Consider the period homomorphism

$$\begin{aligned} \Pi_\omega : H_1(S;\mathbb{Z}) &\longrightarrow \mathbb{C} \\ [\gamma] &\mapsto \int_\gamma \omega, \ \gamma \text{ is a loop.} \end{aligned}$$

This map is well defined. For if $\gamma = \gamma'$ in H_1 then $\gamma - \gamma' = \partial u$, and by the Stokes Theorem we have $\int_\gamma \omega = \int_{\gamma'} \omega$. Also, Π_ω is a $\mathbb{C}$-linear map.

By the de Rham Theorem, we have an isomorphism $H^1_{\text{dR}}(Z) \cong H^1(S;\mathbb{Z}) \otimes \mathbb{C}$ if Z is compact. Such an isomorphism is given by

$$[\omega] \in H^1_{\text{dR}}(Z) \mapsto \Pi_\omega$$

Corollary 2.10.1. $\dim_{\mathbb{C}} H^1_{\text{dR}}(S) = 2g$.

Proof: We have

$$\begin{aligned} 2 - 2g &= \chi(S) \\ &= \sum (-1)^i h^i_{\text{dR}}(S), \text{ where } h^i_{\text{dR}}(S) = \dim(H^1_{\text{dR}}(S)), \\ &= h^0_{\text{dR}}(S) - h^1_{\text{dR}}(S) + h^2_{\text{dR}}(S) \\ &= 1 - \dim_{\mathbb{C}} H^1_{\text{dR}}(S) + 1. \end{aligned}$$

□

Theorem 2.10.1. The de Rham ismomorphism carries wedge product of forms defined by

$$\begin{aligned} H^1_{\text{dR}} \times H^1_{\text{dR}} &\longrightarrow \mathbb{C} \\ ([\omega_1],[\omega_2]) &\mapsto \int_S \omega_1 \wedge \omega_2 \end{aligned}$$

isomorphically to the (cup) product on $H_1(S)$.

Recall that Ω denotes the space of holomorphic 1-forms on S. Since

$$H^1_{\mathrm{dR}} \xrightarrow{\cong} H^1(S;\mathbb{Z}) \otimes \mathbb{C} = \mathrm{Hom}_{\mathbb{C}}(H_1, \mathbb{C})$$

if S is compact, we have an inclusion $\Omega \hookrightarrow H^1_{\mathrm{dR}}$ if S is compact. We have

$$\omega \text{ exact} \implies \omega = df \implies f \text{ holomorphic} \implies f \text{ constant} \implies \omega = 0$$

Definition 2.10.3. $P_g = \dim(\Omega)$ (geometric genus).

It is known that $P_g \geq g$. When the equality holds, it is because of the Hodge Decomposition Theorem.

2.11 Questions about (compact) Riemann surfaces

Recall that if $f : Z_1 \longrightarrow Z_2$ is a finite map of degree n of Riemann surfaces, then any meromorphic function on Z_1 satisfies a polynomial of degree n over the field of Z_2. Hence

$$f^*\mathcal{M}(Z_2) \hookrightarrow f^*(Z_1)$$

is a field extension of degree $\neq n$.

Fact 2.11.1. The equality $\deg = n$ implies that all finite extensions of $\mathcal{M}(Z_2)$ are in natural bijective correspondence with finite maps up to isomorphisms, and each extension is Galois if and only if so is the finite map.

Example 2.11.1. If $Z_2 = \mathbb{P}^1$, $\mathcal{M}(\mathbb{P}^1) = \mathbb{C}(Z)$ then there is a bijective correspondence between fields of transcendence degree $= 1$ and finite covering of $\mathbb{P}^1$.

The equality $\deg = n$ implies that $\mathrm{Aut}(Z) = \mathbb{C}$-Aut of $\mathcal{M}(Z)$.

Question: We know that $2P_g \geq 2g$. When are they equal?

If yes, then a loop C is homologous to 0 if and only if $\int_C \omega = 0$ for every $\omega \in \Omega$. Also, $H^1_{\mathrm{dR}} = \Omega \oplus \overline{\Omega}$. The values $\int_{a_i} \omega$ and $\int_{b_i} \omega$ are called periods.

2.12 Harmonic differentials and Hodge decompositions

Recall that $\mathcal{A}^1 = C^\infty$-valued 1-forms.

Definition 2.12.1. Notice that locally every C^∞-valued 1-form can be written as $\omega = f dz + g d\overline{z}$, where $f dz \in \mathcal{A}^{1,0}$ and $g d\overline{z} \in \mathcal{A}^{0,1}$. There exists a $\mathbb{C}$-linear map $* : \mathcal{A}^1 \longrightarrow \mathcal{A}^1$ called the **star operator**, locally defined by

$$*(pdx + qdy) = -qdx + pdy$$

Or equivalently,

$$*(f dz + g d\overline{z}) = i(-f dz + g d\overline{z}).$$

Every ω is uniquely a sum of $\omega^{1,0} \in \mathcal{A}^{1,0}$ and $\omega^{0,1} \in \mathcal{A}^{0,1}$, and

$$*(\omega) = i(-\omega^{1,0} + \omega^{0,1}).$$

Definition 2.12.2. $\omega \in \mathcal{A}^1$ is **harmonic** if $d\omega = 0 = d(*\omega)$. An 1-form ω is called **coclosed** if $d(*\omega) = 0$. In other words, ω is harmonic if it is closed and coclosed.

Locally, $\omega = f dz + g d\overline{z}$ is closed if $g_z = f_{\overline{z}}$, and is coclosed if $g_z = -f_{\overline{z}}$. Then if ω is harmonic we have $g_z = f_{\overline{z}} = 0$, i.e, $\omega = f dz + g d\overline{z} \in \Omega \oplus \overline{\Omega}$, where $f dz$ is a holomorphic 1-form and $g d\overline{z}$ is an anti-holomorphic 1-form.

Proposition 2.12.1. Let $\mathcal{H}^1$ be the space of harmonic differentials on Z. Then $\mathcal{H}^1 = \Omega \oplus \overline{\Omega}$.

The Hodge Theorem states that $\mathcal{H}^1 = H^1_{\text{dR}}$.

Example 2.12.1. Ω is nonempty for $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z})$.

We have a positive answer to all questions we established.

Theorem 2.12.1 (Hodge Decomposition)**.** $H^1_{\text{dR}}(Z) = \mathcal{H}^1 = \Omega \oplus \overline{\Omega}$, where the first equality is known as the Hodge Theorem.

Theorem 2.12.2 (Riemann Existence Theorem)**.** Let Z be a local coordinate around $p \in Z$, and $n \geq 1$. Then there exists an exact harmonic differential ω on $Z - \{p\}$ such that

$$\omega - d\left(\frac{1}{z^n}\right) = \omega + \frac{n}{z^{n+1}} dz$$

is harmonic on a neighbourhood U of p, and $\omega \in B'_{S-\overline{U}}$, i.e., $\int_{S-\overline{U}} \omega \wedge \overline{\omega} < \infty$.

Corollary 2.12.1.

(1) There exist meromorphic differentials on Z with any preassigned finite set of poles p and any principal parts

$$\omega_p = \sum_{i=-n}^{\infty} a_i z^i dz, \text{ when } n \geq 2.$$

(2) There exist a meromorphic function with any prescribed value at a finite set of points.

(3) $f^*\mathcal{M}_2 \hookrightarrow \mathcal{M}_1$ has degree $= \deg(f)$ for a finite map $f : Z_1 \longrightarrow Z_2$.

2.13 Analysis on the Hilberts space of differentials

There exists a Hermitian inner product for 1-forms ω_1 and ω_2 (at least one which is compactly supported)

$$\boxed{(\omega_1, \omega_2) = \int_Z \omega_1 \wedge *\overline{\omega}_2 < \infty}.$$

Locally, $\omega_i = p_i dx + g_i dy$, with $i = 1, 2$. Then

$$\omega_1 \wedge *\overline{\omega_2} = (p_1\overline{p_2} + g_1\overline{g_2})dx \wedge dy.$$

Hence we can define

$$||\omega||_{L^2}^2 = \int_Z \omega \wedge *\overline{\omega} < \infty.$$

Definition 2.13.1. Let B' be the space of bounded 1-forms ω such that $||\omega||^2 < \infty$.

With respect to the L^2-norm, B^1 is a Hilbert space.

Let E be the closure in B^1 of $d\mathcal{A}_C^0$, where A_C^0 is the space of C^∞-functions with compact support.

Theorem 2.13.1 (Orthogonal decomposition)**.** Let $\omega \in \mathcal{B}^1$. Then there exists a unique orthogonal decomposition

$$\omega = \omega_h + df + *dg$$

where ω_h is bounded harmonic and $f, g \in \mathcal{A}^0$, and $df, dg \in E$.

Proof: The essential point is that the space $\mathcal{H}$ is orthogonal to both E and $*E$, and $E \perp *E$. If $\psi, \varphi \in \mathcal{A}^0$ then

$$\begin{aligned}\left\langle d\varphi, *d\overline{\psi}\right\rangle &= -\int_Z d\varphi \wedge d\psi = \int_Z \psi dd\varphi + \int_Z d(\psi d\varphi) \\ &= 0 + 0.\end{aligned}$$

Similarly, saying that ω is closed means that it is orthogonal to $*E$, and coclosed means that it is orthogonal to E. For example,

$$\begin{aligned}0 = \langle d * \omega, \varphi\rangle &= \int d * \omega \wedge \overline{\varphi} = \int d\overline{\varphi} \wedge *\omega = \langle d\overline{\varphi}, \overline{\omega}\rangle \\ &= \langle \omega, d\varphi\rangle.\end{aligned}$$

Hence

$$\mathcal{H} \oplus_\perp E \oplus_\perp *E \hookrightarrow B^1.$$

To show the equality, we go to the L^2-completion of B' first.

□

Theorem 2.13.2 (Regularity)**.** $\mathcal{H} = (\check{E} \oplus *\check{E})^\perp$ in $\check{B}^1$ where $(^\vee)$ means L^2-completion.

Corollary 2.13.1 (Hodge decomposition)**.** $H^1_{\mathrm{dR}}(Z) = \Omega \oplus \overline{\Omega}$.

Proof: Saying that a form is closed means that it is orthogonal to $*E$. Hence the previous theorem implies that a closed 1-form ω is uniquely $\omega_h + df$, i.e., every element of H^1_{dR} has a unique harmonic representative. $\square$

2.14 Review

Theorem 2.14.1 (Orthogonal decomposition)**.** For every $\omega \in \mathcal{A}^1 =$ space of differntial 1-forms, there exists a unique decomposition

$$\omega = \omega_h + df + *dg$$

where ω_h is a bounded harmonic, $f, g \in \mathcal{A}^0 =$ space of smooth functions.

We denote $\mathcal{A} =$ the space of C^∞-functions.

Proof: Let $\mathcal{H}$ be the space of harmonic differentials. Then $\mathcal{H}$ is orthogonal to both $E = d\mathcal{A}^0$ and $*E = *d\mathcal{A}^0$. This fact follows easily using the L^2-inner product and the equality $** = (-1)^k$, for Riemann surfaces one has $(-1)^k = 1$. We have

$$\mathcal{H} \perp E \perp * E \subseteq \mathcal{A}^1.$$

Taking completion, we have

$$\check{\mathcal{H}} \oplus \check{E} \bigoplus * \check{E} = \overline{\mathcal{A}^1}$$

where $\check{\mathcal{H}} \oplus \check{E}$ is the orthogonal ($\check{\mathcal{H}} \perp \check{E}$) direct sum of $\check{\mathcal{H}}$ and $\check{E}$. By the Weyl's Lemma, we have $\mathcal{H} = \check{\mathcal{H}}$. $\square$

Lemma 2.14.1. Any distribution (1-form) T (1-current) with $\Delta T = 0$ is the distribution of some differential function f, i.e., $T = T_f$ where

$$T_f[h] = \int_U hf$$

and h is compactly supported in $U \subset\subset Z$, where Z is a Riemann surface.

Corollary 2.14.1 (Hodge decomposition)**.** $H^1_{\mathrm{dR}}(Z) = \Omega \oplus \overline{\Omega} = \mathcal{H}$.

2.15 Proof of Weyl's Lemma

Claim 2.15.1. If $f \in \mathcal{A}^0(\mathbb{C})$ then there exists $\psi \in \mathcal{A}^0(\mathbb{C})$ such that $\Delta\psi = f$.

Definition 2.15.1. $\mathcal{A}^0_c(U) = \{f \in C^\infty(U) \ / \ f$ has compact support in $U\}$.

Note that $\mathcal{A}^0_c(U)$ is a topological space of uniform convergence (not complete).

A distribution on U is a continuous linear functional $T : \mathcal{A}^0_c(U) \longrightarrow \mathbb{C}$.

Example 2.15.1. Let $h \in \mathcal{A}^0_c$. Define

$$T_h[f] = \int_U hf$$

Then T is a distribution. Using integration by parts, for $h \in \mathcal{A}^0_c$ and $f \in \mathcal{A}^0_c$, we have

$$\int hD^\alpha f = (-1)^{|\alpha|} \int_Z fD^\alpha h,$$

where $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha = \frac{\partial^{(\alpha_1+\alpha_2)}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$ and $|\alpha| = \alpha_1 + \alpha_2$. In other words, we have

$$T_{D^\alpha h}[f] = (-1)^{|\alpha|} T_h[D^\alpha f]$$

Definition 2.15.2. $(D^\alpha T)[f] = (-1)^{|\alpha|} T[D^\alpha f]$, for every $f \in \mathcal{A}^0_c$.

The definition for $D^\alpha T_h$ is the same as $T_{D^\alpha h}$. Let $g \in \mathcal{A}^0(U \times I)$ where I is an interval, $\text{supp}(g) \subseteq K \times I$ and $K \subset\subset U$.

Let $f_\epsilon(z) = \frac{g(z,t+\epsilon)-g(z,t)}{\epsilon}$. Then $f_\epsilon \longrightarrow \frac{\partial g}{\partial t}$ as $\epsilon \longrightarrow 0$. By continuity of T, we get

$$\frac{d}{dt} T_z(g(z,t)) \overset{\epsilon\to 0}{\longleftarrow} T[f_\epsilon] \overset{\epsilon\to 0}{\longrightarrow} T_z\left[\frac{\partial g(z,t)}{\partial t}\right]$$

So:

Claim 2.15.2. $\frac{d}{dt} T_z[g(z,t)] = T_z\left[\frac{\partial g(z,t)}{\partial t}\right]$.

Similarly, if U and V are open in $\mathbb{C}$ and $K \subset\subset U$, $L \subset\subset V$, $g \in \mathcal{A}^0(U \times V)$ are such that $\text{supp}(g) \subseteq K \times L$, then

$$T_z\left[\int_V g(z,\epsilon)d(\text{Vol}\epsilon)\right] = \int_V T_z(g(z,\epsilon))d\epsilon$$

Let $\rho \in \mathcal{A}^0(\mathbb{C})$ satisfy $\text{supp}(\rho) \subseteq \mathbb{D}$, $\rho(z) = \rho_0(|z|)$ for every z, and $\int_{\mathbb{C}} \rho = 1$. Set $\rho_\epsilon(z) = \frac{1}{\epsilon^2}\rho\left(\frac{z}{\epsilon}\right)$. Then $f \in \mathcal{A}^0(U)$ implies

$$(\rho_\epsilon * f)(z) = \int_U \rho_\epsilon(z-\epsilon)f(\epsilon)d\text{Vol}(\epsilon) \in \mathcal{A}^0(U^{(\epsilon)})$$

where $U^{(\epsilon)} = \{z \in U \ / \ d(z, \partial U) > \epsilon\}$.

Claim 2.15.3. For every $\alpha \in \mathbb{N}^2$, we have

$$D^\alpha(\rho_\epsilon * \rho) = \rho_\epsilon * (D^\alpha f)$$

This result follows similarly applying a change of variables $z \mapsto z + \epsilon$.

Claim 2.15.4. If $z \in U^{(\epsilon)}$ and f is harmonic on $D(z, \epsilon)$, then

$$(\rho_\epsilon * f)(z) = f(z).$$

Proof:

$$(\rho_\epsilon * f)(z) = \int_{D(0,z)} \rho_\epsilon(\epsilon) f(z+\epsilon) d\text{Vol}(\epsilon) = \int_0^{2\pi} \int_0^\epsilon \rho_\epsilon(r) f(z + r\epsilon^{i\theta}) r dr d\theta = 2\pi f(z) \int_0^\epsilon \rho_\epsilon(r) r dr = f(z).$$

□

Proof of Weyl's Lemma: Since $\epsilon \longrightarrow \rho_\epsilon(\epsilon - z)$ has compact support in U, for every $z \in U^{(\epsilon)}$,

$$h(z) := T_\epsilon[\rho_\epsilon(\epsilon - z)]$$

is defined and belongs to $\mathcal{A}^0(U^{(\epsilon)})$ by Claim 6.2. By Claim 6.4, it suffices to show that for every $f \in \mathcal{A}^0_c(U^{(\epsilon)})$ we have

$$T[f] = \int_{U^{(\epsilon)}} hf,$$

where h is harmonic since

$$\Delta h = T_\epsilon[\Delta_z \rho_\epsilon(\epsilon - z)] = \Delta T_\epsilon[\rho(\epsilon - z)] = 0.$$

We want to show that $T = T_h$. We have

$$T[\rho_\epsilon * f] = \int_{U^{(\epsilon)}} hf.$$

Hence it suffices to show

$$T[f] = T[\rho_\epsilon * f].$$

By Claim 6.1, there exists $\psi \in \mathcal{A}^0$ such that $\Delta\psi = f$, where ψ is harmonic on $V = \mathbb{C} - \text{supp}(f)$. Hence $\psi = \rho_\epsilon * \psi$ on V_ϵ by Claim 6.4. Therefore, $\mathcal{O} = \psi - \rho_\epsilon * \rho \in \mathcal{A}^0_c(U)$ satisfies

$$\Delta\mathcal{O} = \Delta\psi - \rho_\epsilon * \Delta\psi = f - \rho * f$$

since $\Delta T = 0$, and $T(\Delta\psi) = 0$. Hence

$$T[f] = T[\rho * f + \Delta\mathcal{O}] = T[\rho_\epsilon * f].$$

□

2.16 Riemann Extension Theorem and Dirichlet Principle

Theorem 2.16.1 (Riemann Extension Theorem)**.** Let Z be a Riemann surface and $p \in Z$. For $n \geq 1$ there exists a harmonic differntial ω on $Z - \{p\}$ such that:

(1) $\omega - d\left(\frac{1}{z^n}\right)$ is harmonic on a small neighbourhood of p.

(2) $\omega \in B'_{Z-\overline{U}}$, i.e., ω is bounded and smooth outside $\overline{U}$, with $||\omega||^2_{L^2} < \infty$.

Outline of the proof: Let $\rho(z)$ be a differentiable function on Z such that $\rho \equiv 0$ outside U and $\rho \equiv 1$ on a neighbourhood of p. The form

$$\psi = d\left(\frac{\rho(z)}{z^n}\right) \in \mathcal{M}(p).$$

We take $p = 0$. Now $\psi - i * (\psi)$ is smooth and has compact support on Z ($\equiv 0$ on a neighbourhood of 0 and outside U), where $*(\psi) = i(udz - vd\overline{z})$ if $\psi = udz + vd\overline{z}$. Hence by the Decomposition Theorem we have

$$\psi - i * (\psi) = \omega_h + df + *(dg)$$

and

$$\omega = \psi - df = \omega_h + i * (\psi) + *(dg)$$

It follows that ω is harmonic ($(d * d)\omega = 0$) since ψ and df are exact.

□

The uniqueness of ω can be guaranteed by adding

(3) $(\omega, dh) = 0$ for every $dh \in \mathcal{A}^1$ such that $||dh|| < \infty$ and $dh \equiv 0$ on a neighbourhood of p.

This condition is the same as the following: Since $dh \equiv 0$ on a neighbourhood N of p, we have

$$\begin{aligned} ||\omega + dh||^2_{Z-N} &= ||\omega||^2_{Z-\overline{N}} + (dh, dh) + (\omega, dh) + \overline{(\omega, dh)} \\ &= ||\omega||^2_{Z-N} + ||dh||^2_{Z-N} \\ &\geq ||\omega||^2_{Z-\overline{N}}. \end{aligned}$$

The Dirichlet Principle states that harmonics ω minimizing $||\ ||^2_{Z-\overline{N}}$ is given by the class of all differentials $\omega + dh$ such that $d\omega = 0$ on N. So, uniqueness is an easy consequence of this.

2.17 Projective model

Theorem 2.17.1. Any compact Riemann surface can be embedded in a projective space.

Proof: $f = (1, f_1, f_2, \ldots, f_n) : Z \longrightarrow \mathbb{CP}^n$ if $f(P) = f(Q)$ for $P \neq Q$. Then just adding a meromorphic function that separates P and Q. This process must terminate by compactness.

□

Theorem 2.17.2 (Chow)**.** The image of f is a projective algebraic variety, i.e., it is $V(P_1, \ldots, P_L)$, for some polynomials $P_1, \ldots, P_l$ on $\mathbb{P}^n$.

Proof: Case $n = 2$: Consider $f : Z \longrightarrow \mathbb{CP}^2$ and $[z_0, z_1, z_2] \in \mathbb{C}^3$. Then $a = f^*\left(\frac{z_0}{z_2}\right)$ and $b = f^*\left(\frac{z_1}{z_2}\right)$ are algebraic dependent over $\mathbb{C}$, i.e., there exists a polynomial $F(Z_1, Z_2)$ of degree d such that $F(a, b) = 0$. Then $x_2^d F\left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right)$ defines $f(Z)$.

□

This result is also known as the 1-st GAGA Principle (after Géométrie Algébrique et Géométrie Analytique).

2.18 Arithmetic nature

Recall that an **(algebraic) number field** F is a finite degree (and hence algebraic) field extension of $\mathbb{Q}$.

An **arithmetic Riemann surface** is a Riemann surface contained in $\mathbb{P}^n$ defined over a number field (i.e., $(P_1, \ldots, P_n) = 0$ defines a Riemann surface and $P_1, \ldots, P_n$ have coefficients in the same number field).

Theorem 2.18.1 (Belgi '79)**.** A Riemann surface Z is arithmetic if and only if there exists a holomorphic map $Z \longrightarrow \mathbb{CP}^1$ with 3 ramification points.

Theorem 2.18.2 (Mardell and Faltings)**.** A Riemann surface Z defined over a number field has a finite number of rational points, i.e., if $g(Z) \geq 2$.

Classification: $\pi_1(Z) = 0$ if and only if the Riemann Mapping Theorem holds, $Z \cong \mathbb{CP}^1, \mathbb{C}$ or $\mathbb{D}$.

Corollary 2.18.1. A Riemann surface Z is rational ($\cong \mathbb{CP}^1$) if and only if $g(Z) = 0$.

Chapter 3

COMPLEX MANIFOLDS

3.1 Complex manifolds and forms

Recall that for a smooth $\mathbb{R}$-manifold M, there is an ideal $\mathcal{I}(x)$ for each $x \in M$, given by

$$\mathcal{I}(x) = \{f \in C^\infty(M) \ / \ f(x) = 0\} \hookrightarrow C^\infty(M)$$

The cotangent plane at $x \in M$ can be defined as the quotient

$$\boxed{T_x^\vee(M) := \mathcal{I}(x)/\mathcal{I}(x)^2}$$

and the tangent plane at $x \in M$ is simply the dual space of the cotangent plane $T_x^\vee(X)$, i.e.,

$$\boxed{T_x(M) := (T_x^\vee(M))^\vee}$$

Definition 3.1.1. An **almost complex structure** on an $\mathbb{R}$-differentiable manifold X of $\dim_{\mathbb{R}} = 2n$ is an epimorphism J of TX such that $J^2 = -1$. Or equivalently, it is the structure of a complex vector bundle on TX.

A complex structure on X induces an almost complex structure on X by setting $J = i = \sqrt{-1}$. We obtain a map $J : T_{X,\mathbb{R}} \longrightarrow T_{X,\mathbb{R}}$ with $\sqrt{-1}$ acting on the domain and J acting in the codomain. We have

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right) \mapsto \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right)$$

Locally, J is defined by

$$\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) \mapsto \left(\frac{\partial}{\partial y_i}, -\frac{\partial}{\partial x_i}\right)$$

Let (X, J) be an almost complex manifold. Then $T_{X,\mathbb{R}} \otimes \mathbb{C}$ contains an eigen-bundle $T_X^{1,0}$ corresponding to the eigen-value i, an an eigen-value $T_X^{0,1}$ corresponding to the eigen-value $-i$, for the operator J. Note that $T_X^{1,0}$ is naturally isomorphic to $T_{X,\mathbb{R}}$ by taking the real part, and this isomorphism identifies i with J. Hence $T_X^{1,0}$ is generated by vectors of the form $u - iJu$, with $u \in T_{X,\mathbb{R}}$.

Theorem 3.1.1. A complex manifold has a complex structure J on $T_{X,\mathbb{R}}$ and its associated subbundle $T_X^{1,0} \subseteq T_{X,\mathbb{R}} \otimes \mathbb{C}$ is naturally the same as T_X (by taking the real part).

Similarly, if $\Omega_{X,\mathbb{R}} := T^{\vee}_{X,\mathbb{R}}$, then

$$\Omega_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega_X \oplus \overline{\Omega}_X$$

with the identification

$$\Omega_X \longleftrightarrow \Omega_X^{1,0} \longleftrightarrow dz_i,$$
$$\overline{\Omega}_X \longleftrightarrow \Omega_X^{0,1} \longleftrightarrow d\overline{z}_i.$$

Definition 3.1.2. An almost complex structure is called **integrable** if it comes from a complex structure.

The only spheres with an almost complex structure are S^2 and S^6. The 6-sphere S^6 has an almost complex structure via the octonions, by taking the multiplication structure in the multiplicative structure in the sphere in the purely imaginary part of octonions.

Question: Does it exist complex structures on S^2?

Theorem 3.1.2 (Newlande - Niremberg)**.** J is integrable if and only if $[T_X^{1,0}, T_X^{1,0}] \subseteq T_X^{1,0}$.

Proposition 3.1.1 (Poincaré Lemma)**.** Let α be a closed differential from a differentiable manifold with $\deg(\alpha) > 0$. Then locally $\alpha = d\beta$, for some form β.

Proposition 3.1.2 ($\overline{\partial}$-Poincaré Lemma)**.** Let α be a $\overline{\partial}$-closed differential from a differentiable manifold with $(p,q) = \deg(\alpha)$ and $q > 0$. Then locally $\alpha = \overline{\partial}\beta$, for some form β.

Proof: First we show that we can reduce the problem to the case $p = 0$ and $q = 1$. In general, we know

$$\alpha = \sum \alpha_{I,J} dz^I \wedge dz^J$$
$$\overline{\partial}\alpha = \sum d\alpha_{I,J} \wedge dz^I \wedge d\overline{z}^J = 0$$

Then $\alpha_I = \sum_{\alpha_{I,J}} \overline{\partial} z^J$ is $\overline{\partial}$-closed and of type $(0,q)$, and so $\alpha_I = \overline{\partial}\beta$ if and only if $\alpha = (-1)^p \overline{\partial}(\sum \partial z^I \wedge \beta_I)$. Henceforth assume that α is of type $(0,q)$, i.e., $\alpha = \sum \alpha_J d\overline{z}^J$. Apply the induction on the largest integer k such that $k \in J$ and $\alpha_J \neq 0$. Necessarily $k \geq q$ and $k = q$ implies $\alpha = f d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_q$. In the latter case $\overline{\partial}\alpha = 0$ if and only if f is holomorphic in the variables z_i with $i > q$. We may know apply the following result:

Proposition: There exists a differentiable function g holomorphic in the variables z_i, with $i > q$, such that $\frac{\partial g}{\partial \overline{z}_q} = f$ and hence $\alpha = (-1)^{q-1}\overline{\partial}(g d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_{q-1})$.

Now assume the $\overline{\partial}$-Poincaré Lemma proved for $k - 1 > q$. Write $\alpha = \alpha_1 + \alpha_2 d\overline{z}_k$, $\alpha_2 = \sum \alpha_{2,J} d\overline{z}^J$ where $|J| = q - 1$ and $J \subseteq \{1, \ldots, k-1\}$. So $\overline{\partial} = 0$ implies $\alpha_{2,J}$ is holomorphic in variables z_l, with $l > k$. Hence by the previous proposition, we have $\alpha_{2,J} = \frac{\partial \beta_{2,J}}{\partial \overline{z}_k}$, where $\beta_{2,J}$ is holomorphic in z_l, $l > k$. Then

$$\overline{\partial}\beta = \overline{\partial}(\beta_{2,J} d\overline{z}^J) = (-1)^{q-1}\alpha_2 \wedge d\overline{z}_k + \alpha_1'$$

where α_1' involves only the coordinate z_l for $l < k$. Thus,

$$\alpha = \alpha_1'' + \overline{\partial}\beta$$

where α_1'' for $l < k$. Since β is holomorphic in the z_l for $l > k$, we have $\overline{\partial}\alpha_1'' = 0$, $q = \deg(\alpha_1'') < k$. We conclude by induction.

□

Proof of the previous proposition: We restrict to the case $p = 0$ and $q = 1$. Let $\alpha = f d\overline{z}$. As statement is local, we may assume that $\mathrm{supp}(f)$ is compact. Define

$$g = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(z)}{\zeta - z} d\zeta \wedge d\overline{\zeta} := \lim_{\epsilon \to 0} \frac{1}{2\pi i}\int_{\mathbb{C}\setminus D(z,\epsilon)} \frac{f(z)}{\epsilon - z} dz \wedge d\overline{z}.$$

This limit exists since f is bounded and $\frac{1}{\zeta - z}$ in integrable on $\mathbb{D}$. We want to show that $\overline{\partial}g = \alpha = f d\overline{z}$. Now $g = g_\epsilon$ where $g_\epsilon = \frac{1}{2\pi i}\int_{\mathbb{C}\setminus D(0,\epsilon)} \frac{f(\gamma + z)}{\gamma} d\gamma \wedge d\overline{\gamma}$. Then

$$\overline{\partial}g_\epsilon(z) = \frac{1}{2\pi i}\left(\int_{\mathbb{C}\setminus D(0,\epsilon)} \partial_{\overline{z}} f(\gamma + z)\frac{d\gamma \wedge d\overline{\gamma}}{\gamma}\right) d\overline{z}$$

implies

$$\begin{aligned}\frac{\partial}{\partial \overline{z}} g(z) = \partial_{\overline{z}} g(z) &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial}{\partial \overline{z}} f(\gamma + z)\frac{d\gamma \wedge d\overline{\gamma}}{\gamma} \\ &= \lim_{\epsilon \to 0} \frac{1}{2\pi i}\int_{\mathbb{C}\setminus D(z,\epsilon)} \frac{\partial}{\partial \overline{\zeta}} f(\zeta)\frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z}.\end{aligned}$$

On $\mathcal{C}\setminus D(z,\epsilon)$, we have $\frac{\partial f}{\partial \overline{\zeta}}(z)\frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z} = -d_\epsilon\left(f(\zeta)\frac{d\zeta}{\zeta - z}\right)$. By the Stokes Theorem, we get

$$\frac{1}{2\pi i}\int_{\mathbb{C}\setminus D(z,\epsilon)} \frac{\partial f}{\partial \overline{\zeta}}\frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D(z,\epsilon)} \frac{f(\zeta)}{\zeta - z} \longrightarrow f(z) \text{ as } \epsilon \longrightarrow 0.$$

Hence $\overline{\partial}g = f d\overline{z}$.

□

3.2 Kähler manifolds

Let V be a complex vector space with $J = \sqrt{-1}$ and $W = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Recall $V_{\mathbb{C}} := V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$. Hence also $W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1} \supseteq W$.

Definition 3.2.1. Let $W^{1,1} = W^{1,0} \otimes W^{0,1} \subseteq \Delta^2 W_{\mathbb{C}} \supseteq \Delta^2 W$,

$$W^{1,1} = \{(1,1)\text{-forms}\} = \{\text{sesqui-linear forms on } V\}.$$

Let $W^{1,1}_{\mathbb{R}} = W^{1,1} \cap \Delta^2 W = \{\text{real } (1,1)\text{-forms}\} = \{\text{real 2-forms of type } (1,1)\} = \{\text{alternating forms}\}$. A $(1,1)$-form $h \in W^{1,1}$ is called **Hermitian** if $h(u,v) = \overline{h(v,u)}$ for every $u, v \in V$. Let $W^{1,1}_H$ be the space of such forms.

Fact 3.2.1. There exists a bijective correspondence between Hermitian forms and real alternating forms of type $(1,1)$ via

$$W^{1,1}_H \ni h \longleftrightarrow \mathrm{Im}(h) \in W^{1,1}_{\mathbb{R}}$$

Proof: Since $h(u,v) = \overline{h(v,u)}$, we have that $\mathrm{Im}(h)$ is alternating on V, i.e., $\mathrm{Im}(h) \in \Delta^2 W$. Conversely, let $\omega \in W^{1,1}_{\mathbb{R}}$ and set

$$g(u,v) = \omega(u, Jv) = -\omega(Ju, v) \text{ and}$$
$$h(u,v) = g(u,v) - i\omega(u,v).$$

Then $g(u,v) = g(v,u)$ and thus $h(u,v) = \overline{h(v,u)}$, i.e., h is Hermitian.

□

Locally, $\omega = \sum \frac{i}{2} a_{ij} dz_i \wedge d\bar{z}_j = -\mathrm{Im}(h) \in \Omega^{1,1}_X \cap \Omega^2_{X,\mathbb{R}}$, where (a_{ij}) is hermitian.

Definition 3.2.2. $\omega \in W^{1,1}_{\mathbb{R}}$ is **positive** if the correspondence h is positive definite.

Definition 3.2.3. A **positive real** $(1,1)$-form on an almost complex manifold (X, J) is a C^∞ associated of a positive real $(1,1)$-form on each tangent space $T_{X,x}$, $x \in X$.

Definition 3.2.4. A **Hermitian metric** on a complex vector bundle E over a smooth manifold M is an element $h \in \Gamma(E \otimes \overline{E})^*$. A **Hermitian manifold** is a complex manifold with a Hermitian metric on its holomorphic tangent space. Likewise, an **almost Hermitian manifold** is an almost complex manifold with a Hermitian metric on its holomorphic tangent space.

Corollary 3.2.1. There exists a bijective correspondence between real $(1,1)$-forms ω on a complex manifold M and Hermitian metrics on M.

Definition 3.2.5. Let h be a Hermitian metric. We shall say that h is **Kähler** if $\omega = \mathrm{Im(h)}$ is closed.

Corollary 3.2.2. If a symplectic structure ω on a complex manifold is positive of type $(1,1)$ (i.e., it vanishes on $\Omega^{2,0}$ and hence also on $\Omega^{0,2}$ and its associated h is positive definite), then it is $-\mathrm{Im}(h)$ for a Kähler metric.

Corollary 3.2.3. A Hermitian metric can always be written as $h = g + i\omega$ where g is a Riemannian metric invariant under J $(g(u,v) = g(Ju, Ju))$ and ω is a positive $(1,1)$-form, $g(u,v) = \omega(u, Jv)$.

Definition 3.2.6. A pair (X, ω) formed by a complex manifold X and a positive $(1,1)$-form ω is called a **Kähler manifold**.

Lemma 3.2.1. $d\mathrm{Vol}_h = \frac{\omega^n}{n!}$ for (X,h), $h = h_\omega = g(u,v)$, where $\omega^n = \omega \wedge \cdots \wedge \omega$ of type (n,n).

Proof: Let $\{e_i\}$ be an orthonormal basis of $T_{X,x}$ with respect to h. Then $\{e_i, Je_i\}$ is a real orthonormal basis for $T^{\mathbb{R}}_{X,x}$ with respect to g with positive orientation. It suffices to check $\frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ where $\{dx_1, dy_1, \ldots, dx_n, dy_n\}$ is the dual basis to (e_i, Je_i). Let $dz_j = dx_j + dy_j$. Then we have $\omega_x = \frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$ and

$$\frac{\omega_x^n}{n!} = \left(\frac{i}{2}\right)^n \prod_j dz_j \wedge d\bar{z}_j \text{ at } x,$$

$$\frac{i}{2} dz_j \wedge d\bar{z}_j = dx_i \wedge dy_j.$$

□

Corollary 3.2.4. If $X^{(n)}$ is a compact Kähler manifold then $[\omega^k] \in H^{2k}_{\mathrm{dR}}(X)$ is nonzero for every $k < n$.

Proof: $\omega^k = d\gamma$ implies $\omega^n = d(\omega^{n-k} \wedge \gamma)$. The last implies $0 \neq \int_X \omega^n = 0$, getting a contradiction. □

Corollary 3.2.5. Let $X^{(k)}$ be a compact Kähler submanifold M. Then $[x] \in H^{2k}(X)$ is nonzero.

Proof: Clearly $h_M|_{TX} = h_X$ and $i^*\omega_{(M,h)} = \omega_{(X,h_X)}$, where $i : X \hookrightarrow M$ is the inclusion. If $i(X) = \partial\Gamma$, then by the Stokes Theorem

$$\mathrm{Volume}_X = \int_X i^*\omega_M^h = \int_\Gamma d\omega_M^h = 0 \text{ since } d\omega_M^h \equiv 0.$$

□

3.3 Metrics and connections

Let $E \longrightarrow X$ be a C^∞-vector bundle on X, and let $\mathcal{A}^i(E)$ be the vector space of C^∞ E-valued forms on X.

Definition 3.3.1. A real (complex) **connection** on E is a real (resp. complex) linear map

$$\nabla : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^1(E)$$

satisfying the Leibniz rule:

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

For a vector field ψ and $\sigma \in \mathcal{A}^0(E)$ we write

$$\nabla_\psi \sigma = (\nabla\sigma)(\psi) \in \mathcal{A}^0(E).$$

In the case where E is a holomorphic vector bundle, we have the operation

$$\overline{\partial}_E : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{0,1}(E)$$

which defines holomorphic sections of E via $\mathrm{Ker}(\overline{\partial}_E)$. It satisfies the $\overline{\partial}$-Leibniz rule instead:

$$\overline{\partial}(f\sigma) = \overline{\partial}f \otimes \sigma + f\overline{\partial}\sigma$$

but it is not a complex connection.

Proposition 3.3.1 (For a Riemannian manifold)**.** If (M, g) is a $\mathbb{R}$-manifold then there exists a unique connection ∇ on TM called the **Levi-Civita connection** satisfying:

(1) $d(g(\psi_1, \psi_2)) = g(\psi_1, \nabla\psi_2) + g(\nabla\psi_1, \psi_2)$, i.e., g is ∇-invariant.

(2) $\nabla_{\psi_1}\psi_2 - \nabla_{\psi_2}\psi_1 = [\psi_1, \psi_2]$, i.e., g is torsion free of ∇.

Theorem 3.3.1 (and definition)**.** Let $E \longrightarrow X$ be a holomorphic vector bundle with a Hermitian metric. There exists a unique complex connection ∇ on E, called the **Chern connection** satisfying:

(1) $d(h(\sigma, \tau)) = h(\nabla\sigma, \tau) + h(\sigma, \nabla\tau)$, i.e., ∇ is invariant under (or compatible with) h.

(2) Let $\nabla^{0,1}$ be its composition with $\mathcal{A}^1(E) \longrightarrow \mathcal{A}^{0,1}(E)$. Then $\nabla^{0,1} = \overline{\partial}_E$.

Theorem 3.3.2. The following statements for a complex Hermitian manifold (X, h) are equivalent:

(1) h is Kähler.

(2) J is flat for the Levi-Civita connection.

(3) Chern connection = Levi-Civita connection.

3.4 Review

A complex structure on a real manifold M of dimension $2n$ is an endomorphism J of TM such that $J^2 = -1$. If M is complex, normally we take $J = \sqrt{-1}$. A real 2-form h is Hermitian if $h(u,v) = \overline{h(v,u)}$. It is know that a form h is Hermitian if and only if h is a positive $(1,1)$-form.

Theorem 3.4.1. There exists a bijective correspondence between real alternating forms of type $(1,1)$ and Hermitian metrics. Such a correspondence is given by

$$W_H^{1,1} \xrightarrow{\sim} W_{\mathbb{R}}^{1,1}$$
$$h \mapsto \mathrm{Im}(h)$$

where $\mathrm{Im}(h)$ is a symplectic 2-form.

Theorem 3.4.2. The following conditions are equivalent for a complex Hermitian manifold (X,h):

(i) h is a Kähler metric, i.e., $dw_h = 0$.

(ii) J is flat for the Levi-Civita connection of h.

(iii) The Chern connection of h on $T_M^{1,0}$ equals the Levi-Civita connection on $T_M^{\mathbb{R}}$.

Proof:

- (iii) $\Longrightarrow$ (ii): It is clear because the Chern connection is $\mathbb{C}$-linear by definition.
- (ii) $\Longrightarrow$ (i): Condition (ii) means that the Levi-Civita connection commutes with J. Then

$$d\omega(\varphi_1,\varphi_2) = \omega(\nabla\varphi_1,\varphi_2) + \omega(\varphi_1,\nabla\varphi_2).$$

  Let $C^\infty(M) \ni \varphi[\omega(\varphi_1,\varphi_2)] = \omega(\nabla_\varphi\varphi_1,\varphi_2) + \omega(\varphi_1,\nabla_\varphi\varphi_2)$. Since

$$d\omega(\varphi,\varphi_1,\varphi_2) = \varphi\omega(\varphi_1,\varphi_2) - \varphi_1\omega(\varphi,\varphi_2) + \varphi_2\omega(\varphi_1,\varphi) - \omega([\varphi,\varphi_1],\varphi_2)$$

  the result follows from $[\varphi_i,\varphi_j] = \nabla_{\varphi_1}\varphi_j - \nabla_{\varphi_2}\varphi_1$.
- (i) $\Longrightarrow$ (iii): The Chern connction equals the Levi-Civita connection for the flat metric $\sum_i dz_i \wedge d\overline{z}_i$. The result follows from the following proposition.

□

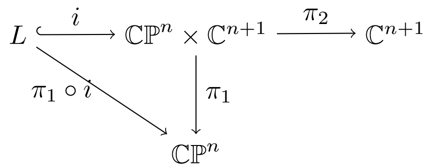
Proposition 3.4.1. If (X,h) is a Kähler manifold and if $x \in X$, then there exists a holomorphic coordinate $(z_1,\ldots,z_n)$ centred at x such that

$$h_{ij} = h\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right) = \mathrm{Im} + O(\sum |z_i|^2).$$

The converse is also true.

3.5 The Fubini Study metric

Let $L = \{(l, v) \in \mathbb{CP}^n \times \mathbb{C}^n \ / \ v \in l\}$. Consider the diagram



The composite map $\pi_2 \circ i$ is called the **blow up** at 0. We have that L is a holomorphic line bundle over $\mathbb{CP}^n$ and is denoted by $O(-1)$.

Definition 3.5.1. $O_{\mathbb{CP}^n}(h) := L^{-k}$ where $L^{-k} := (L^{\vee})^{\otimes k}$ for $k > 0$, is a holomorphic line bundle over $\mathbb{CP}^1$.

The standard metric $\sum |z_i|^2$ on $\mathbb{CP}^{n+1}$ restricts to a Hermitian metric on L. Its curvature (Ricci or Chern form) is given by

$$\omega = \sigma^* \frac{2}{2\pi} \partial\overline{\partial} \log|z_i|^i = \frac{i}{2\pi} \partial\overline{\partial} \log|\sigma|^2$$

for any choice of a holomorphic section σ of L over $\mathbb{CP}^n$. Therefore, $\sigma' = \sigma f$, for $f \in \mathcal{O}$ and so

$$\log(\sigma')^2 = \log|\sigma|^2 + \log|f|^2,$$

where $\log|f|^2 = \log f + \log \overline{f}$ and it is $\partial\overline{\partial}$-closed.

Lemma 3.5.1. ω is a positive $(1,1)$-form.

Proof: We prove only the case $n = 1$. We have

$$\overline{\partial} \log(1 + |z|^2) = \frac{\overline{\partial}(1 + |z|^2)}{1 + |z|^2} = \frac{z d\overline{z}}{1 + |z|^2}.$$

So

$$\omega = \frac{i}{2\pi} \frac{[(1 + |z|^2) dz \wedge d\overline{z} - \overline{z} dz \wedge z d\overline{z}]}{(1 + |z|^2)^2} = \frac{i}{2\pi} \frac{dz \wedge d\overline{z}}{(1 + |z|^2)^2}$$

and the conclusion follows from the transitivity of $SU(n+1)$ on $T\mathbb{CP}^n$.

□

Definition 3.5.2. ω is called the **Fubini study metric** in $\mathbb{CP}^n$ and is denoted ω_{FS}. It depends on the choice of coordinates on $\mathbb{C}^{n+1}$.

Similarly for a holomorphic vector bundle $E \longrightarrow X$ with a Hermit metric h, one has the line bundle

$$\begin{array}{ccccc} L & \overset{i}{\hookrightarrow} & \pi^{-1}(E) & \overset{\tilde{\pi}}{\longrightarrow} & E \\ & \searrow & \downarrow & & \downarrow \Pi \\ & & \mathbb{P}(E) & \overset{\pi}{\longrightarrow} & X \end{array}$$

where $\mathbb{P}(E) = (E \backslash \{\text{zero sections}\})/\mathbb{C}^*$ and L is denoted by $L = O_{\mathbb{P}(E)}(-1)$. The composition $\overline{\pi} \circ i$ is called the **blow up** of E at its zero section. Here $\pi^{-1}(E)$ is the fibre product or pullback of π and Π. Let $F = (\mathbb{P}E)_{x \in X}$ be the fibre of π at x and $f : F \hookrightarrow \mathbb{P}E$ the inclusion. Then $f^* c_1(|\ |_h^2)$ is a positive $(1,1)$-form, where $c_1(|\ |_h^2) = \frac{i}{2\pi}\partial\overline{\partial}\log|\ |_h^2$. Hence $c_1(|\ |_h^2)$ is a $(1,1)$-form on $\mathbb{P}(E)$ that is positive in the vertical direction of π of X, where X is Kähler with Kähler form ω_X. Hence $\mathbb{P}(E)$ is also Kähler.

Definition 3.5.3. $O_{E_\varphi}(h) := L^{-k}$ where $L^{-k} := (L^\vee)^{\otimes k}$.

Note that given a vector bundle $E \xrightarrow{\varphi} X$, then $1_\varphi = \varphi_{E_\varphi}(1) = L^\vee = L^{-1}$ is a holomorphic line bundle over $\mathbb{P}(E)$.

Consider the compactification $\overline{E} = \mathbb{P}(E \oplus O) \supseteq E$, $E \xrightarrow{\varphi} X$ and $E \oplus O \xrightarrow{\overline{\varphi}} X$ are vector bundles over X, and E is open in $\mathbb{P}(E \oplus O)$. We see that the blow up of E (or $\overline{E}$) along its zero section lies in $\mathbb{P}(\overline{\varphi}^{-1}(E))$ and hence it is Kähler.

Chapter 4

SHEAF COHOMOLOGY

4.1 Sheaves

Definition 4.1.1. Let X be a topological space and $\mathcal{A}$ an abelian category. A **presheaf** $\mathcal{F}$ on X is a collection of objects $\mathcal{F}(U)$ of objects in $\mathcal{A}$, for each open subset $U \subseteq X$, and a collection of morphisms

$$\begin{aligned} \rho_{UV} : \mathcal{F}(U) &\longrightarrow \mathcal{F}(V) \\ \sigma &\mapsto \sigma|_V = \rho_{UV}(\sigma) \end{aligned}$$

for each inclusion of open subsets $V \hookrightarrow U$ such that

$$\rho_{UV} = \rho_{VW} \circ \rho_{UV}.$$

The last equality is known as **compatibility**.

A presheaf $\mathcal{F}$ is called a **sheaf** if it is **saturated**, i.e., if it satisfies the following condition: Let $s_i \in \mathcal{F}(U_i)$ be a collection of sections such that

$$s_i|_{U_{ij}} = s_j|_{U_{ij}},$$

where $U_{ij} = U_i \cap U_j$, then there exists a unique section $s \in \mathcal{F}(\cup U_i)$ such that $s|_{U_i} = s_i$.

Definition 4.1.2. A **morphism of (pre)sheaves** is a map $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ which associates to each open subset $U \subseteq X$ a morphism

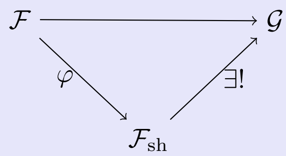
$$\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

such that for every $V \subseteq U$ open

$$\rho_{UV} \circ \varphi_U = \varphi_V \circ \rho_{UV}. \text{ (compatibility)}$$

Example 4.1.1. Sheaves of sections of vector bundles (C^∞, C^h, C^ω for real analytic, $\mathcal{O}$, etc).

Lemma 4.1.1. If $\mathcal{F}$ is a presheaf over X then there exists a unique sheaf $\mathcal{F}_{\text{sh}}$ along with a morphism $\mathcal{F} \longrightarrow \mathcal{F}_{\text{sh}}$ that factors thorough all morphisms $\mathcal{F} \longrightarrow \mathcal{G}$ to a sheaf $\mathcal{G}$.



If X is a topological space, its structure sheaf $\mathcal{C}^0_X$ associated to each open subset U is given by the space of continuous functions on U. If X is an algebraic variety, its structure sheaf $\mathcal{O}_X$ is given by the space of regular functions on (Zariski) open subsets. A complex algebraic manifold is also a complex manifold, and we write $\mathcal{O}^{\text{alg}}_X$ and $\mathcal{O}^{\text{hol}}_X$ to distinguish the structures:

$$\mathcal{O}^{\text{alg}}_X \longrightarrow \text{ Zariski topology,}$$
$$\mathcal{O}^{\text{hol}}_X \longrightarrow \text{ ordinary topology.}$$

Definition 4.1.3. Let $\mathcal{A}$ be the sheaf of rings over X. A sheaf $\mathcal{F}$ is called an $\mathcal{A}$-module if for every open set U, $\mathcal{F}(U)$ is a module over $\mathcal{A}(U)$, compatible with the restriction maps.

Remark 4.1.1. All notions from module theory carry over: Hom, Ker, Im, CoKer, direct sums, tensor products, exact sequences and homology groups, etc.

Definition 4.1.4. The $\mathcal{A}$-module $\mathcal{A}^{\oplus n} = \mathcal{A}$ is said to be **free of rank** $n \in \mathbb{N}$. A sheaf that is locally isomorphic to $\mathcal{A}^{\oplus n}$ is called **locally free** of rank n.

Remark 4.1.2. There exists a bijection between vector bundles of rank n and locally free sheaves of rank n.

If $\mathcal{A}$ is a sheaf of fields, the rank one sheaves (invertible sheaves with respect to $\mathcal{A}$) form a group under tensor product, called the **Picard group** $\text{Pic}_{\mathcal{O}_X}(X)$.

Definition 4.1.5. Let $(f, f^{\#}) : (X, \mathcal{A}) \longrightarrow (Y, \mathcal{B})$ be a morphism of ringed spaces, i.e., $f : X \longrightarrow Y$ is a continuous function, and

$$f^{\#}_{:}\mathcal{B}(V) \longrightarrow \mathcal{A}(f^{-1}(V))$$

is a morphism for every open subset V compatible with restrictions. The **pullback sheaf** $f^*\mathcal{G}$ of a $\mathcal{B}$-module $\mathcal{G}$ is defined as follows: set $f^{(*)}\mathcal{G}(U) = \varinjlim_{f(U)\hookrightarrow V} \mathcal{G}(V)$ and then set $f^*\mathcal{G}$ be the sheaf associated to the presheaf $f^{(*)}\mathcal{G} \otimes_{f^{(*)}\mathcal{B}} \mathcal{A}$.

Example 4.1.2. The sheaf $\mathcal{F}$ of holomorphic sections of a holomorphic vector bundles F over X,

$$i_X : \{x\} \hookrightarrow X$$

(1) $i^{(*)}_x \mathcal{F} =: \mathcal{F}_x$ (the **stalk** of $\mathcal{F}$ at x and its elements are germs of sections of F at x).

(2) $i_x^*\mathcal{F} = F_x$ has finite dimension over $\mathbb{C}$.

Definition 4.1.6. A sheaf of modules $\mathcal{M}$ on an algebraic variety $(X, \mathcal{O}_X)$ is said to be **(quasi)-coherent** if it is locally isomorphic to the cokernel of a morphism of free sheaves (of finite rank).

Easy fact: f^* preserves (locally) free sheaves, rank and invertibility. In particular, f^* gives a homomorphism of Picard groups.

Definition 4.1.7. A short sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is said to be **exact** if and only if it is exact on the level of stalks.

Let $(f, f^*) : (X, \mathcal{A}) \longrightarrow (Y, \mathcal{B})$ be a morphism of ringed spaces, i.e., $f : X \longrightarrow Y$ is a continuous map, $\mathcal{A}$ and $\mathcal{B}$ are shaves of rings. For a sheaf of $\mathcal{A}$-modules $\mathcal{F}$ on X, the **direct image sheaf** $f_*\mathcal{F}$ of $\mathcal{F}$ on X is the sheaf of $\mathcal{B}$-modules on Y given by $V \mapsto \mathcal{F}(f^{-1}(V))$. Recall that for a sheaf of $\mathcal{B}$-modules $\mathcal{G}$ on Y, its pullback is defined as follows: Set $f^{(*)}\mathcal{G}(U) = \varinjlim_{f(U)\subseteq V} \mathcal{G}(V)$, and then set $(f^*\mathcal{G})$ to be the sheaf associated with the presheaf $f^{(*)}\mathcal{G} \otimes_{f^{(*)}\mathcal{B}} \mathcal{A}$.

Example 4.1.3. Let $\mathcal{F}$ be the sheaf of holomorphic sections of a vector bundle F and $i : \{x\} \longrightarrow (X, \mathcal{O})$.

(1) $i^{(*)}\mathcal{F} =: \mathcal{F}_x$ is called the stalk of i, and it equals the set of germs of sections of F.

(2) $i^*\mathcal{F} = F_x$, the fibre of F at x, is a finite dimensional vector space.

Definition 4.1.8. Recall that a sheaf of modules on an algebraic variety $(X, \mathcal{O}_X)$ is said to be **quasi-coherent** (resp. **coherent**) if it is locally isomorphic to the cokernel of a morphism of free shaves (resp. of finite rank). By a free sheaf we mean a sheaf $\mathcal{O}_X^{\oplus n}$, where n is a cardinal number.

Fact 4.1.1. f^* preserves local freeness invertibility, in particular f^* gives a homomorphism of Picard groups, where

$$\text{Pic}(X) = \text{ group of invertible sheaves } \cong \text{ holomorphic line bundles}$$

A short sequence of shaves $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is said to be exact if it is exact at the level of stalks.

Remark 4.1.3. Ker, CoKer, $\otimes$ and f^* preserve the property of being (quasi-)coherent. However, f_* does not.

Example 4.1.4.

(1) $f : \mathbb{C} \longrightarrow \{0\}$. Then $f_*\mathcal{O}_{\mathbb{C}}^{\text{alg}} = \mathbb{C}[z]$ which is not finite dimensional.

(2) $i : \mathbb{C}^* \longrightarrow \mathbb{C}$. Then $i_*\mathcal{O}_{\mathbb{C}^*}^{\text{alg}} = \mathbb{C}[z, z^{-1}]$ is not of finite type over $\mathbb{C}[z]$.

Construction: Let X be an affine variety over $\mathbb{K}$, and M a module over $\mathbb{K}[x]$. Then

$$X = \text{ zeroes of } \{f_1, \dots, f_l\} \text{ over } \mathbb{C}^N.$$

We have

$$\mathbb{K}[X] := \mathbb{C}[z_1, \dots, z_N] / \langle f_1, \dots, f_l \rangle$$

Then $U \mapsto M \otimes_{\mathbb{K}[X]} \mathcal{O}_X(U)$ is an $\mathcal{O}_X$-module and this correspondence preserves tensor product, exactness, etc. Call this $\mathcal{O}_X$-module $\tilde{M}$. In part, $\tilde{M}$ is quasi-coherent (and coherent if M is of finite type) and any quasi-coherent $\mathcal{O}_X$-module is of this form.

Example 4.1.5 (Important)**. Ideal sheaf** $\mathcal{I}_Y \hookrightarrow \mathcal{O}_X$ corresponding to the subsheaf of $\mathcal{O}_X$ vanishing on $Y \hookrightarrow X$ (i.e., Y is an algebraic subvariety).

We normally assume that the ground field $\mathbb{K}$ is algebraically closed. Then the Nullstellensatz tells us that $V(\mathcal{I}) := \sup(\mathcal{O}_X/\mathcal{I}) \subseteq X$ is nonempty if and only if $\mathcal{I} \neq \mathcal{O}_X$. Hence the ringed space $(V(\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ is identified with the **subscheme** of X corresponding to $\mathcal{I}$. A ringed space locally isomorphic to subschemes of affine spaces is called al **algebraic scheme**.

Definition 4.1.9 (Associated fibre spaces)**.** Let $\mathcal{A}$ be a quasi-coherent sheaf of a $\mathcal{O}_X$-algebra of finite type (i.e., locally generated by finitely many sections as $\mathcal{O}_X$-algebras). We define a **scheme** $S = \text{Specm}_X \mathcal{A}$ and a morphism $\pi : S \longrightarrow (X, \mathcal{O}_X)$ as follows: Let X be affine. Then set

$$\text{Specm}_X \mathcal{A} := \text{Specm}\mathcal{A}(X) := \{\text{maximal ideals in } \mathcal{A}(X)\}.$$

Recall that if $R = \mathcal{A}(X)$ and $\mathcal{M}$ is a maximal ideal, then $R/\mathcal{M} = \mathcal{K}$ if $\mathcal{K}$ is algebraically closed.

Let $f : S \longrightarrow \mathbb{K}$ be a regular function. Then $\{f = 0\}^c =$ a basis of open sets, form the Zariski topology. Let π be the dual to the $\mathbb{K}$-algebra homomorphism $\mathbb{K}[X] \longrightarrow \mathcal{A}(X)$. If $D(f) = \{x \in X \ / \ f(x) \neq 0\}$ for $f \in \mathcal{O}_X(X)$ then by the quasi-coherence of $\mathcal{A}$, we have $\mathcal{A}(D(f)) = \mathcal{A}(X) \otimes_{\mathbb{K}[X]} \mathbb{K}[D(f)]$. So that $\text{Specm}\mathcal{A}(D(f)) = \pi^{-1}(D(f))$.

Special case: Given a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, let $\mathcal{A} = \text{Sym}_{\mathcal{O}_X} \mathcal{F}$. Then the associated fibre space is called the **vector fibre space**, denoted by $\pi : \mathbb{V}(\mathcal{F}) \longrightarrow X$. Here, Sym means the symmetric product $\oplus_{k \geq 0} (\text{Sym}^k \mathcal{F})$. Note that

$$\mathbb{V}(\mathcal{F})_x = (\mathcal{F}_x / \mathcal{M}_x \mathcal{F}_x)^\vee$$

Example 4.1.6 (for algebraic geometry)**.**

(I) Definition of **normal** and **tangent** bundles (cones):

- The model for tangent vector space at a point is given by $\text{Specm}(\mathbb{K}[\epsilon]/\epsilon^2)$.
- The **Zariski tangent space**: Let $x \in X$ be a point of al algebraic variety. Then $T_x X := (\mathcal{M}_x / \mathcal{M}_x^2)^\vee$.
- The **normal bundle** of $Y \hookrightarrow X$ (algebraic subvariety): Let $\mathcal{I}$ be the ideal sheaf of Y, then $\mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. The normal vector bundle (or normal bundle) of Y in X is $\mathcal{I}/\mathcal{I}^{\not{2}}$. It is denoted by $N_{Y|X} (\xrightarrow{\pi} Y)$.

– The **normal cone** of Y in X is defined by

$$\begin{array}{ccc} C_{Y|X} := \operatorname{Specm}_Y \left(\oplus_{k \geq 0} \mathcal{I}^k / \mathcal{I}^{k+1} \right) & \hookrightarrow & N_{Y|X} \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

An easiest definition of the tangent bundle to an algebraic variety X is that it is the normal cone to the diagonal $X \overset{\Delta}{\hookrightarrow} X \times X$. These are functorial objects since $f : X \longrightarrow Y$, $f \times f : X \times X \longrightarrow Y \times Y$, so $T_f : TX \longrightarrow TY$. And it coincides with $d_x f : T_x X \longrightarrow T_{f(x)} Y$, for every $x \in X$.

– We say that $x \in X$ is a **smooth point** if $C_x(X) = T_x X$, and X is **smooth** (**non-singular**) if all points are.

(II) The **cotangent sheaf** to X is defined as the conormal sheaf to the diagonal in $X \times X$. Its local sections are local forms on TX and such a form d gives a map

$$\mathcal{M}_x / \mathcal{M}_x^2 \overset{\sim}{\longrightarrow} \Omega'_X(x) := \Omega'_{X,x} / (\Omega'_{X,x} \otimes \mathcal{M}_x)$$

where $\Omega'_X = \{$ differential on $\mathcal{O}_X \}$.

(III) **Blowing up a subscheme**: Let $\mathcal{I} \hookrightarrow \mathcal{O}_X$ be an ideal sheaf defining a subscheme $Y \hookrightarrow X$, $\mathcal{A} = \oplus_{k>0} \mathcal{I}^k$. Then $\sigma : \tilde{X} = \operatorname{proj}(\mathcal{A}) \longrightarrow X$ is called the blow up of X along Y, where $\operatorname{proj}(\mathcal{A}) =$ Specm(homogeneous decomposition of $\mathcal{A}$). By functoriality, $\sigma^{-1}(Y)$ is the projection of the algebra $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \oplus_{k \geq 0} \mathcal{I}^k / \mathcal{I}^{k+1}$, i.e., $\sigma^{-1}(Y) \longrightarrow Y$ is the projectivization of the normal cone $C_{Y|N}$, i.e., $0 = \otimes \mathcal{O}_X$.

Definition 4.1.10. A sheaf is torsion free is $\otimes \mathcal{O}_X = 0$, i.e., it is supported on a subvariety.

Fact 4.1.2.

- Any torsion free $\mathcal{O}_X$-module $\mathcal{F}$ admits a resolution, i.e., a birational morphism $\sigma : Y \longrightarrow X$ such that $\sigma^* \mathcal{F}$ is locally free.
- (Hironaka) Any variety X (any rational map $X \longrightarrow Y$) admits a resolution of singularities by repeatedly blow ups along smooth centres (i.e., smooth subvariety)

$$\begin{array}{ccc} \tilde{X} & \overset{\sigma}{\longrightarrow} & X \\ & \searrow & \vdots \\ & & Y \end{array}$$

4.2 Cohomology of sheaves

Let X be a topological space. We consider sheaves $\mathcal{F}^i$ together with morphisms $d_i : \mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$ such that $d_{i+1} \circ d_i = 0$ for every i. Such set of sheaves and morphisms $(\mathcal{F}_i, d_i)$ is called a **complex of sheaves over** X. It is also called a **resolution** of a sheaf $\mathcal{F}$ if there exists an inclusion $\iota : \mathcal{F} \longrightarrow \mathcal{F}_0$ such that $i(\mathcal{F}) = \mathrm{Ker}(d_0)$ and $\mathrm{Ker}(d_{i+1}) = \mathrm{Im}(d_i)$, for every i.

(1) **Čech resolution:** Let $\mathcal{F}$ be a sheaf over X, $\{U_i\}_{i\in\mathbb{N}}$ a covering of X. For every finite subset $I \subseteq \mathbb{N}$ set $U_I = \cap_{i\in I} U_i$, $j_I : U_I \hookrightarrow X$ and

$$\mathcal{F}_I = (j_I)_*(\mathcal{F}|_{U_I}) \text{ (extension by zero outside } U_I)$$

Define $\mathcal{F}^k := \oplus_{|I|=k+i} \mathcal{F}_I$ and $d : \mathcal{F}^k \longrightarrow \mathcal{F}^{k+1}$ by

$$(d\sigma)_{j_0 \dots j_{k+1}} = \sum_i (-1)^i \sigma_{j_0} \dots \hat{j}_i \dots j_{k+1}|_{U \cap U_I}$$

where $j_0 \leq j_1 \leq \dots \leq j_{k+1}$, $\sigma = (\sigma_I)$, $\sigma_I \in \mathcal{F}_I(U)$ and $|I| = k+1$. Lastly, we define $\iota : \mathcal{F} \longrightarrow \mathcal{F}^0$ by

$$\iota(\sigma)_i = \sigma|_{U \cap U_i} \text{ for } \sigma \in \mathcal{F}(U).$$

Proposition 4.2.1. This is a resolution.

(2) **de Rham resolution:** Let $\mathcal{A}^k$ be the sheaf of C^∞ ($\mathbb{R}$ or $\mathcal{C}$-valued) differential forms of degree k (on a real or complex manifold). The d-Poincaré Lemma says that the complex $(\mathcal{A}^k, d)$ is a resolution of $\mathrm{Ker}(d_0) = \underline{\mathbb{R}}$ (resp. $\underline{\mathbb{C}}$), constant sheaves over X.

(3) **Dolbeault resolution:** Let E be a holomorphic vector bundle over a complex manifold X and $\mathcal{E}$ its sheaf of holomorphic sections (i.e., $\mathcal{E} = \mathcal{O}_X(E)$). Let $\mathcal{A}^{0,q}$ be the sheaf of C^∞-sections of $\Omega_X^{0,q} \otimes E$. Generalizing the $\overline{d}$-Poincaré Lemma, we get that $(\mathcal{A}^{0,q}(E), \overline{\partial})$ is a resolution of $\mathrm{Ker}(\overline{\partial_0}) = \mathcal{E} = \mathcal{O}_X(E)$ (a coherent $\mathcal{O}_X$-module).

4.3 Coherent sheaves

Let $\mathcal{O}_X$ be the space of functions on X. The "sheaf version"of this space means that to every open subset $U \subseteq X$ it is associated an space of forms on U.

Example 4.3.1. To an algebraic variety X we associate the space of rational 1, 1-forms on X without poles on U. To any complex manifold X we associate the space of holomorphic functions on U.

A coherent sheaf is free if it is of the form $\mathcal{O}_X^{\oplus n}$. An ideal sheaf $\mathcal{I}$ is just a subsheaf of $\mathcal{O}_X$ which is an ideal of $\mathcal{O}_X(U)$ (a ring) for every U. Normally assume $\mathcal{I} \neq \mathcal{O}_X$. We have a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{Z(\mathcal{I})} \longrightarrow 0,$$

where $Z(\mathcal{I})$ is a scheme equal to $Z(\mathcal{I}) = \mathrm{spec}(\mathcal{O}_X/\mathcal{I}) \subsetneq X$, which is nonempty. Note that $\mathcal{O}_{Z(\mathcal{I})}$ and $\mathcal{I}$ are examples of coherent sheaves over X, and $\mathcal{O}_{Z(\mathcal{I})}$ is called the **torsion part**. Any coherent sheaf is locally a finite direct sum of these factors. If there are no factors of the form $\mathcal{O}_{Z(\mathcal{I})}$, i.e., not supported on a proper subvariety, then it is called **torsion free**. Any torsion free sheaf admits a resolution $f : \widetilde{X} \xrightarrow{\text{res}} X$ such that f^* of the sheaf is locally free (i.e., a vector bundle). This is because given an ideal sheaf $\mathcal{I}$ defining a subscheme $Y \subseteq X$, $A = \oplus_{k>0}\mathcal{I}^k$ is an algebra over $\mathcal{O}_X$. Then $\sigma : X = \mathrm{proj}(A) \longrightarrow X$ (an isomorphism outside Y), called the **blowup** of X along Y, is a birational map to X that replaces Y by a subscheme of codimension 1, i.e., $\sigma^{-1}(Y)$ is locally given by one equation and $\sigma^{-1}(Y) \longrightarrow Y$ is the projectivization of the normal cone $C_{Y|X}$.

Given a collection of sheaves $\mathcal{F}^i$ over X with morphism $d_i : \mathcal{F}^i \longrightarrow \mathcal{F}^{i+1}$ such that $d_{i+1} \circ d_i = 0$ for every i. It is a **resolution** of a sheaf $\mathcal{F}$ if there exists an inclusion $i : \mathcal{F} \longrightarrow \mathcal{F}_0$ such that $j(\mathcal{F}) = \mathrm{Ker}(d_0)$ and $\mathrm{Ker}(d_{i+1}) = \mathrm{Im}(d_i)$ for every i.

(1) There exists a sheaf X, $\{U_i\}_{i\in\mathbb{N}}$ covering of X. For every finite set $I \subseteq \mathbb{N}$, set $U_I = \cap_{i\in I} U_i$, $j_I : U_I \hookrightarrow X$, and $\mathcal{F}_I = (j_I)_*(\mathcal{F}_{U_I})$ extended by zero outside U_I. Set $\mathcal{F}^k = \oplus_{|I|=k+1}\mathcal{F}_I$ and $d : \mathcal{F}^k \longrightarrow \mathcal{F}^{k+1}$ by

$$(d\sigma)_{j_0\cdots j_{k+1}} = \sum_i (-1)^i (\sigma_{j_0\cdots\hat{j}_i\cdots j_{k+1}})|_{U\cap U_I},$$

where $\sigma = (\sigma_I)$, $\sigma_I \in \mathcal{F}_I^k(U)$, and $j : \mathcal{F} \longrightarrow \mathcal{F}^0$ is given by $j(\sigma)_i = \sigma|_{U_i\cap U}$ for $\sigma \in \mathcal{F}(U)$.

Proposition 4.3.1. This is a resolution, where $\mathcal{F}_i|_{U_i}$ has trivial cohomology.

(2) **de Rham resolution:** Let $\mathcal{A}^k$ be the sheaf of C^∞ ($\mathbb{R}$ or $\mathbb{C}$)-valued k-forms. The Poincaré Lemma says that the complex $(\mathcal{A}^k, d)$ is a resolution of $\mathrm{Ker}(d_0) = \mathbb{R}$ or $\mathbb{C}$, constant sheaves.

(3) Let E be a holomorphic vector bundle and $\mathcal{E}$ its sheaf of holomorphic sections. Let $\mathcal{A}^{0,q}(E)$ be the sheaf of C^∞-sections $\Omega_X^{0,q} \otimes E$. The $\overline{\partial}$-Poincaré Lemma implies that $(\mathcal{A}^{0,q}(E), \overline{\partial})$ is a resolution of $\mathrm{Ker}(\overline{\partial_0}) = \mathcal{E}$.

4.4 Derived functors

Given a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ between abelian categories such that $\mathcal{C}$ has enough injectives.

Definition 4.4.1. For every object $M \in \mathcal{C}$, there exists an object $R^i F(M)$ for every $i \geq 0$ in $\mathcal{C}'$, unique up to isomorphisms, satisfying the following conditions:

(1) $R^0 F(M) = F(M)$,

(2) Every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ gives rise to a long exact sequence in $\mathcal{C}'$

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow R^1 F(A) \longrightarrow \cdots$$
$$\cdots \longrightarrow R^i F(A) \longrightarrow R^i F(B) \longrightarrow R^i F(C) \longrightarrow R^{i+1} F(A) \longrightarrow \cdots$$

Remark 4.4.1.

(1) Let $\mathcal{M}^0$, $i : \mathcal{A} \longrightarrow \mathcal{M}^0$ be an **acyclic resolution** of $\mathcal{A}$ (i.e., $R^{j+1} F(\mathcal{M}^k) = 0$ for every $j, k \geq 0$). Then
$$R^j F(A) = H^j(F(\mathcal{M}^0)) := \mathrm{CoKer}(d^{i-1} \mathcal{F}(\mathcal{M}^{i-1}) \longrightarrow \mathcal{F}(\mathcal{M}^i))$$

(2) For a sheaf $\mathcal{F}$ over X there exists an acyclic resolution (called the **Godement resolution** for $\mathcal{F}$).

Definition 4.4.2. A **fine sheaf** $\mathcal{F}$ os a sheaf of $\mathcal{A}$-modules where the sheaf of algebra $\mathcal{A}$ admits a partition of unity: for every open covering $\{U_i\}$, there exists $f_i \in \mathcal{F}(U_i)$ such that $\sum_i f_i = 1$ (this sum is locally finite).

Proposition 4.4.1. $H^i(X, \mathcal{F}) = 0$ for every $i \geq 0$ for such $\mathcal{F}$.

Corollary 4.4.1.

(1) Let X be a real (complex) manifold, then

$$H^*(X; \mathbb{R}) = \frac{\mathrm{Ker}(d)}{\mathrm{Im}(d)} = H^*_{\mathrm{dR}}(X; \mathbb{R}),$$

where the same equality holds if we replace $\mathbb{R}$ by $\mathbb{C}$.

(2) Given a holomorphic vector bundle $E \longrightarrow X$ and $\mathcal{E}$ its sheaf of holomorphic sections. Then

$$H^*(X, \mathcal{E}) = \frac{\mathrm{Ker}(\overline{\partial} : \mathcal{A}^{0,q}(E) \longrightarrow \mathcal{A}^{0,q+1}(E))}{\mathrm{Im}(\overline{\partial} : \mathcal{A}^{0,q-1}(E) \longrightarrow \mathcal{A}^{0,q}(E))}$$

Corollary 4.4.2 (Grothendieck Vanishing Theorem)**.** Given a holomorphic vector bundle $E \longrightarrow X$, we have $H^q(X, E) = 0$ for $q > n = \dim_{\mathbb{C}} X$.

Consider a Čech resolution

$$\mathrm{Ker}(d) = \mathcal{F} \longrightarrow \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \longrightarrow \cdots \longrightarrow \mathcal{F}^k \longrightarrow \cdots$$

where the map $\mathcal{F} \longrightarrow \mathcal{F}^0$ is given by $\sigma \mapsto \sigma|_{U \cap U_i}$ and $\mathcal{F}^k = \oplus_{|I|=k+1} (j_I)_* (\mathcal{F}|_{U_I})$.

Theorem 4.4.1. If $H^{q>0}(U_I, \mathcal{F}) = 0$ for every $I \subseteq \mathbb{N}$ then

$$H^q(X, \mathcal{F}) = \check{H}^q(U, \mathcal{F}) := H^q(\Gamma(\mathcal{F}) = \mathcal{F}(X), d_X),$$

e.g., if X is a finite dimensional manifold C^∞, then there exists a good cover (U_I contractible for every I) and so

$$\check{H}^q(U, \mathbb{Z}) = H^q_{\text{cell}}(X, \mathbb{Z})$$

where U is in an open cover and $H^q_{\text{cell}}(X, \mathbb{Z})$ denotes the cellular cohomology, which is computing via nerves of the open covering.

Chapter 5

HARMONIC FORMS

5.1 Harmonic forms on compact manifolds

Let (X, g) be a Riemannian manifold, where X compact is a blanket assumption. Then we have a metric $(\ ,\ )$ on $\Delta^* T^\vee_{X,x}$. We assume X is oriented. Let $\alpha, \beta \in \mathcal{A}^k : C^\infty(\mathcal{A}^k T^\vee_X)$. Then

$$\langle \alpha, \beta \rangle = \int_X \langle \alpha, \beta \rangle_x \, d\mathrm{Vol}(x)$$

gives an L^2-metric on $\mathcal{A}^k$. We also have a pointwise isomorphism $p : \Delta^{n-k} T^\vee_x \xrightarrow{\sim} \mathrm{Hom}(\Delta^k T^\vee_x, \Delta^n T^\vee_x)$ given by $v \mapsto v \wedge -$, where $\Delta^n T^\vee_x = \mathbb{R} d\mathrm{Vol}(x)$, and an isomorphism $m : \Delta^k T^\vee_x \xrightarrow{\sim} \mathrm{Hom}(\Delta^k T^\vee, \mathbb{R})$ given by $e \mapsto \langle e, \ \rangle_{\Delta^k T^\vee}$.

Definition 5.1.1. The **Hodge Star Operator** is given by

$$* = p^{-1} \circ m : \Delta^k T^\vee_x \xrightarrow{\sim} \Delta^{n-k} T^\vee_x$$

and the associated global isomorphism by

$$\begin{gathered} * : \Delta^k T^\vee \xrightarrow{\sim} \Delta^{n-k} T^\vee \\ \Omega^k(X) \longrightarrow \Omega^{n-k}(X) \end{gathered}$$

We extend $*$ to complex-valued forms by extending $\langle\ ,\ \rangle$ to Hermit metrics on $\Delta^k_{\mathbb{C}}(T^\vee \otimes \mathbb{C}) = (\Delta^k_{\mathbb{R}} T^\vee) \otimes \mathbb{C}$. We get $(\alpha, \beta)_x d\mathrm{Vol}(x) = \alpha_x \wedge \overline{\beta_x}$ and so

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge * \overline{\beta}$$

is the $\mathcal{L}^2$-metric on $\mathcal{A}^k_{\mathbb{C}} = \mathcal{A}^k \otimes \mathbb{C}$. In the case X is complex, $\mathcal{A}^k_{\mathbb{C}} = \oplus_{p+q=k} \oplus \mathcal{A}^{p,q}$, $\mathcal{A}^{p,q} = C^\infty(\Omega^{p,q}_X)$.

Fact 5.1.1. The Stokes Theorem implies that $(\alpha, d^*\beta) = (d\alpha, \beta)$ where $d^* := (-1)^k *^{-1} d *$.

In part, if n is even then $d^* = - * d*$. Similarly,

Fact 5.1.2. $\partial^* = - * \overline{\partial} *$ and $\overline{\partial}^* = - * \partial *$ are formal adjoint of ∂ and $\overline{\partial}$ with respect to the $\mathcal{L}^2$ metric in $\mathcal{A}^k_{\mathbb{C}}$.

Proof: $(\overline{\partial}\alpha, \beta) = \int_X \overline{\partial}\alpha \wedge *\overline{\beta} = -\int_X (-1)^{|\alpha|}\alpha \wedge \overline{\partial} * \overline{\beta} = -\int_X (-1)^{|\alpha|}\alpha \wedge \overline{* *^{-1} \partial * \beta} = (\alpha, \overline{\partial}^* \beta)$. □

More generally, if (E, h) is a Hermitian vector bundle then there exists a $\mathbb{C}$-anti linear isomorphism of vector bundles given by $h : \Omega^{0,q}_X \otimes E = \Omega^{0,q}(E) \longrightarrow (\Omega^{0,q} \otimes E)^\vee \cong \Omega^{n,n-q} \otimes E^\vee$, where $\Delta^{2n}_X = \Omega^{n,n} = \mathbb{R}d\text{Vol}(x)$. So it gives am antilinear isomorphism

$$*_E : \Omega^{0,q}(E) \xrightarrow{\sim} \Omega^{n,n-q}(E^\vee) = K_X \otimes \Omega^{0,n-q}(E^\vee)$$

called the **Hodge Star**, where $K_X = \Omega^{n,0} = \Delta^n_{\mathbb{C}} T^\vee_X$ (holomorphic line bundle) is called the **canonical bundle** of a complex manifold X.

Fact 5.1.3. $\overline{\partial}^*_X = (-1)^q *^{-1}_E \circ \overline{\partial}_{K_X \otimes E^\vee} : \mathcal{A}^{0,q}(E) \longrightarrow \mathcal{A}^{0,q-1}(E)$ is the formal adjoint of $\overline{\partial}_E$.

Fact 5.1.4. $(d^*)^2 = (\overline{\partial}^*_E)^2 = (\partial^*)^2 = 0$.

Definition 5.1.2. Let (X, g) be a Riemannian manifold,

$$\Delta := dd^* + d^*d = (d + d^*)^2$$

Definition 5.1.3. Let (X, h) be a Hermitian manifold,

$$\Delta_\partial := \partial\partial^* + \partial^*\partial = (\partial + \partial^*)^2,$$
$$\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} = (\overline{\partial} + \overline{\partial}^*)^2.$$

If further $E \longrightarrow X$ is a holomorphic vector bundle with a Hermit metric, we write Δ_E for $\Delta_{\overline{\partial}_E} = (\overline{\partial}_E + \overline{\partial}^*_E)^2$. From construction,

$$\langle \alpha, \Delta_d \alpha \rangle = ||d\alpha||^2 + ||d^*\alpha||^2$$

and analogously for the Hermitian case.

Corollary 5.1.1. $\text{Ker}(\Delta_d) = \text{Ker}(d) \cap \text{Ker}(d^*)$.

Definition 5.1.4. An element of $\text{Ker}(\Delta_d)$ is called **harmonic**, i.e., it is killed by d and d^*.

Theorem 5.1.1 (Main Theorem of the Course)**.** Let

$$(F,\phi) = \begin{cases} \left(\oplus_k \Omega_X^k, \Delta_d\right) & \text{for the Riemannian case,} \\ \left(\oplus_q \Omega^{0,q}(E), \Delta_{\overline{\partial}_E}\right) & \text{for the Hermitian case.} \end{cases}$$

where $\phi : F \longrightarrow F$ is an automorphism, and $F = \oplus_k \Omega_X^k$ or $F = \oplus_q \Omega^{0,q}(E)$.

5.2 Some applications of the Main Theorem

Theorem 5.2.1 (Riemannian case)**.** Let $\mathcal{H}^k$ be the space of harmonic k-forms. Then the map $H^k \longrightarrow H^k_{\mathrm{dR}}(X, \mathbb{R} \text{ (or } \mathbb{C}))$ given by $\alpha \mapsto [\alpha]$, which is well defined since $d\alpha = 0$, is an isomorphism.

Proof: By the Main Theorem, $\beta \in \mathcal{A}^k$ can be written as $\alpha + \Delta\gamma = \alpha + dd^*\gamma + d^*d\gamma$, where $\alpha + dd^*\alpha$ is d-closed. Since β is closed we have $d^*d\gamma$ is closed and hence it belongs to $(\mathrm{Im}(d^*))^{\perp}$. So $0 = (d\gamma, dd^*d\gamma) = ||d^*d\gamma||^2$. Similarly, we get the analogous theorem for $\Delta_{\overline{\partial}_E}$ where we can identify $H^q(X, \mathcal{O}(E) = \mathcal{E})$ with $H^{0,q}_{\overline{\partial}}(X, E)$ via the Dolbeaut isomorphism.

□

Theorem 5.2.2. Given a holomorphic vector bundle $E \longrightarrow X$, let $\mathcal{H}^{0,q}(E)$ be the space of $(\overline{\partial}_E)$-holomorphic forms of type $(0, q)$ with values in E. Then the map $\mathcal{H}^{0,q}(E) \longrightarrow \mathcal{H}^q(X, \mathcal{E})$ given by $\alpha \mapsto [\alpha]$ is an isomorphism.

Corollary 5.2.1. If X be a compact manifold then $H^q(X, \mathbb{R})$ is finite dimensional. If X is a compact complex manifold and $E \longrightarrow X$ is a holomorphic vector bundle then $H^q(X, E)$ is finite dimensional.

5.3 Review

Theorem 5.3.1. Let X be a compact manifold:

$$(F,P) = \begin{cases} \left(\bigoplus_k \Omega_X^k, \Delta_d\right) & \text{if } X \text{ is Riemannian,} \\ \left(\bigoplus_q \Omega^{0,q}(E), \Delta_{\overline{\partial}_E}\right) & E \longrightarrow X \text{ is a holomorphic vector bundle} \end{cases}$$

where $P$is an elliptic or Laplacian-like operator. This implies that

$$C^\infty(F) = \operatorname{Ker}(P) \bigoplus P(C^\infty(F))$$

where $\operatorname{Ker}(P)$ is finite dimensional.

Corollary 5.3.1.

(1) There is an isomorphism $\mathcal{H}^k \xrightarrow{\sim} H^k_{\mathrm{dR}}(X, \mathbb{R}\text{or } \mathbb{C})$ given by $\alpha \mapsto [\alpha]$, where $\mathcal{H}^k$ is the finite dimensional space of harmonic forms.

(2) There is an isomorphism $\mathcal{H}^{0,q} \xrightarrow{\sim} H^q(X, E)$, where $\mathcal{H}^{0,q}$ is the finite dimensional space of $(0,q)$-forms.

Both isomorphisms depend on the chosen metric.

Note that

$$\begin{array}{ccc} H^k_{\mathrm{dR}}(X,\mathbb{R}) & \xrightarrow{\cong^1} & H^k(X, \underline{\mathbb{R}}) \\ & \searrow^{\cong^2} \quad {}^{\cong^3}\swarrow & \\ & \check{H}(H,\mathbb{R}) & \end{array}$$

where $\cong^1$ is given by the de Rham isomorphism, $\cong^2$ and $\cong^3$ are given by the Poincaré Lemma. Recall the Dolbeault isomorphism:

$$H^{p,q}_{\overline{\partial}}(X,E) \cong H^{0,q}_{\overline{\partial}}(X, \Omega^p E) \cong^{(*)} H^{p,q}(X, \Omega^p E) =: H^{p,q}(X,E) \text{ (finite dimensional),}$$
$$\Omega^p E \cong \mathcal{O}(\Delta^p T_X^* \otimes E) \text{ (space of holomorphic } p\text{-forms with values in } E).$$

where $(*)$ comes from the isomorphism $\Omega^p E = \operatorname{Ker}\overline{\partial^0}(\mathcal{A}^{p,0})(E) \longrightarrow \mathcal{A}^{p,1}(E))$. Also,

$$H^{p,q}(X) = H^{p,q}(X, \underline{\mathbb{C}}) = H^q(X, \Omega^p_X) \text{ (sheaf of holomorphic functions).}$$

It is known that $\Delta = \partial \Delta_{\overline{\partial}}$ in every Kähler manifold. Then the following theorem follows:

Theorem 5.3.2. Let X be a Kähler manifold. Then

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

In other words, if $[\alpha] \in H^k(X,\mathbb{C})$ with α harmonic, then $\alpha = \sum_{p+q=k} \alpha^{p,q}$, where $\alpha^{p,q}$ is the (harmonic) component of type (p,q).

5.4 Heat equation approach

Given an initial distribution of heat $f(x) = F(x,0)$ $(t=0)$ on a Riemannian manifold (X,g), then the heat $F(x,t)$ at time t is governed by $(\partial_t + \Delta_X)F = 0$.

Example 5.4.1. F is easily obtained for every t for S^1, and in general for the torus as follows:

$$F(0,t) = \sum a_n(t)e^{in\theta}.$$

We have

$$\begin{aligned}\partial_t + \Delta_\theta = 0 &\Longrightarrow 0 = \sum \left(a_n'(t) + n^2 a_n(t)\right) e^{in\theta}\\ &\Longrightarrow a_n(t) = a_n e^{-n^2 t}, \text{ where } a_n = a_n(0),\\ &\Longrightarrow F(\theta,t) = \sum_{n\geq 0} e^{-n^2 t} a_n e^{in\theta}.\end{aligned}$$

It follows $F(\theta,t) \longrightarrow a_0 = \int_{S^1} f(\theta)d\theta = \mathrm{Av}_{S^1}(f)$, where the integral is the initial distribution.

Example 5.4.2. Let $X = \mathbb{R}$. Doing the same exercise as above but using Fourier transforms, we get

$$F(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)dy = \int_{\mathbb{R}} e_{\mathbb{R}}(x,y,t)f(y)dy.$$

The function $e_{\mathbb{R}}(x,y,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}$ is called the **heat kernel**. Similarly, $e_{\mathbb{R}^n}(x,y,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{||x-y||^2}{4t}}$. On S^1, we have $e(x,y,t) = \sum_n e^{-n^2 t}e^{in(x-y)} = \sum_n e^{-n^2 t}e^{inx}\overline{e^{iny}}$, where $e^{-n^2 t}$ are the eigenvalues of Δ, and e^{inx} and $\overline{e^{iny}}$ are the eigenfunctions. In general, the existence of $e_X(x,y,t)$ is difficult to obtain analytically but trivial on physical grounds.

Remark 5.4.1. $F(x,t)$ is smooth for every $t > 0$, i.e., immediate smoothing by heat flow.

In general, given a form α on (X,g), wish to solve

$$(*)\begin{cases} (\partial_t + \Delta)A(t) = 0\\ A(0) = \alpha\end{cases}$$

where $\alpha(t)$ is a form on X parametrized by t. Uniqueness of $A(t)$ follows from:

Lemma 5.4.1. $||A(t)||$ is decreasing (non-strict) for a solution of $(*)$.

Proof: $\partial_t ||A(t)||^2 = 2\langle \partial_t A, A\rangle = -2\langle \Delta A, A\rangle = -2\left\langle ||dA||^2 + ||d^* A||^2\right\rangle \leq 0.$ □

Theorem 5.4.1. Let (X,g) be a compact Riemannian manifold. Then there exists $K_p(x,y,t) \in \mathcal{A}^p_x(X)$, depending only on (X,g) and p, called the **heat kernel** of X on p-forms such that

$$A(t) = \int_X K_p(\cdot, y, t)\alpha(y) d\mathrm{Vol}(y)$$

solves $(*)$, for every $\alpha \in \mathcal{A}^p(X)$.

Let $T_t(x) = \int_X K(\cdot, y, t)\alpha(y)dy$.

Theorem 5.4.2. T_t satisfies:

(1) $T_{t_1+t_2} = T_{t_1}T_{t_2}$.

(2) T_t is formally self-adjoint.

(3) $T_t\alpha$ tends to a C^∞ harmonic form $H(\alpha)$ as $t \longrightarrow \infty$.

(4) $G(\alpha) = \int_0^\infty (T_t\alpha - H\alpha)dt$ is well defined and yields the **Green operator** G, i.e.

$$G(\alpha) \perp \text{(harmonic forms)} \;\text{ and }\; \alpha = H(\alpha) + \Delta G(\alpha).$$

Proof:

(1) Holds because $A(t_1 + t)$ solves the heat equation with initial condition $A(t_1)$.

(2) $\partial_t \langle T_t\eta, T_\tau\epsilon\rangle = \langle \partial_t T_t\eta, T_\tau\epsilon\rangle = -\langle \Delta T_t\eta, T_\tau\epsilon\rangle = -\langle T_t\eta, \Delta T_\tau\epsilon\rangle = \langle T_t\eta, \partial_t T_\tau\epsilon\rangle = \partial_t \langle T_t\eta, T_\tau\epsilon\rangle$. This implies that $\langle\ ,\ \rangle$ is a function of $t+\tau$, so denote $\langle\ ,\ \rangle$ by $g(t+\tau)$. Therefore,

$$\langle T_t\eta, \epsilon\rangle = g(t+0) = g(0+t) = \langle \eta, T_\tau\epsilon\rangle .$$

(3) (1) + (2) $\Longrightarrow \forall h > 0$,

$$\begin{aligned} ||T_{t+2h}\alpha - T_t\alpha||^2 &= ||T_{t+2h}\alpha||^2 + ||T_t\alpha||^2 - 2\langle T_{t+2h}\alpha, T_t\alpha\rangle \\ &= ||T_{t+2h}\alpha||^2 - ||T_t\alpha||^2 - 2(||T_{t+h}\alpha||^2 - ||T_{t+2h}\alpha||||T_t\alpha||) \end{aligned}$$

and $||T_\alpha\alpha||^2$ converges, and therefore it is decreasing. Hence $||T_{t+2h}\alpha - T_t\alpha||^2 \longrightarrow 0$. It follows $T_t\alpha \longrightarrow H(\alpha)$, for some $H(\alpha) \in \overline{\mathcal{A}^p(X)}^{L^2}$, called the **harmonic projection**. Fix $\tau > 0$, then $T_t\alpha = T_\tau T_{t-\tau}\alpha \longrightarrow H(\alpha) := T_\tau H(\alpha)$ as $t \longrightarrow \infty$. Hence $H(\alpha)$ is C^∞ since T_τ is given by a C^∞ kernel. Hence $H = \lim_{t\to\infty} T_t$ is also formally self-adjoint.

(4) $||T_t\alpha - H\alpha||$ can be shown to decay rapidly enough so that $G(\alpha) = \int_0^\infty (T_t\alpha - H(\alpha))dt$ is well defined. We verify that G is formally the Green operator:

$$\Delta G(\alpha) = \int_0^\infty \Delta T_t\alpha dt = \int_0^\infty -\partial_t T_t\alpha dt = \alpha - H(\alpha),$$

and for β harmonic we have

$$\langle G\alpha, \beta\rangle = \int_0^\infty \langle (T_t - H)\alpha, \beta\rangle\, dt = \int_0^\infty \langle \alpha, (T_t - H)\beta\rangle\, dt = 0, \text{ since } (T_t - H)\beta = 0.$$

□

Corollary 5.4.1. There exists an orthogonal direct sum decomposition

$$\mathcal{A}^p(X) = \mathcal{H}^p(X) \bigoplus d(\mathcal{A}^{p-1}(X)) = \mathcal{H}^p(X) \bigoplus d^*(\mathcal{A}^{p-1}(X))$$

where $\mathrm{Im}(\Delta) = \mathrm{Im}(d) + \mathrm{Im}(d^*)$.

Corollary 5.4.2. There is an isomorphism $\mathcal{H}^p(X) \cong H^p_{\mathrm{dR}}(X)$ given by $\alpha \mapsto [\alpha]$.

Then $\mathcal{O} \in \mathcal{A}^p(X) \implies \mathcal{O} = \alpha + dd^*\gamma + d^*d\gamma$, where $\alpha \in \mathcal{H}(X)$. Also,

$$||d^*d\gamma||^2 = \langle d^*d\gamma, \alpha \rangle = \langle d\gamma, d\alpha \rangle \text{, where } d\alpha = 0.$$

5.5 Index Theorem (Heat Equation approach)

Let $\Delta^p : \mathcal{A}^p(X) \longrightarrow \mathcal{A}^p(X)$, $\lambda \in \mathbb{R}_{\geq 0}$. Let E^p_λ denote the λ-eigenspace for Δ^p (finite dimensonal). The square root $\sqrt{\Delta} = \delta$ is called the **Dirac operator**.

Lemma 5.5.1. The sequence

$$0 \longrightarrow E^0_\lambda \xrightarrow{d} E^1_\lambda \longrightarrow \cdots \xrightarrow{d} E^n_\lambda \longrightarrow 0$$

is exact for $\lambda > 0$.

Proof: $\omega \in E^p_\lambda \implies \Delta^{p+1} d\omega = d\Delta^p\omega = \lambda d\omega \implies d\omega \in E^{p+1}_\lambda$.
Now $\omega \in E^p_\lambda$ and $d\omega = 0 \implies \lambda\omega = \Delta^p\omega = d^*d + dd^*\omega \implies \omega = d\left(\frac{1}{\lambda}d^*\omega\right)$. Then $\Delta d^*\omega = d^*\Delta\omega = \lambda d^*\omega$. □

Corollary 5.5.1. $\sum_p (-1)^p \dim(E^p_\lambda) = 0$.

Corollary 5.5.2. Let $\{\lambda^p_i\}$ be the spectrum of Δ^p, with terms repeated n times if multi $= n$. Then

$$\sum_p (-1)^p \mathrm{tr} e^{-t\Delta^p} = \sum_p (-1)^p e^{-t\lambda_i^{(p)}} = \sum_p (-1)^p \sum_i{}' e^{-\lambda_i^{(p)}(t)}$$

where $\sum_i'$ is over i where $\lambda_i^{(p)} = 0$.

Note that $\sum_i' e^{-\lambda_i^{(p)}(t)} = \dim(\mathrm{Ker}(\Delta^p))$. Hence

$$\begin{aligned}\mathcal{X}(X) = \sum_p (-1)^p \dim(\mathrm{Ker}(\Delta^p)) &= \sum_p (-1)^p \mathrm{tr} e^{-t\Delta(p)}, \text{ where } e^{-t\Delta(p)} = T_t,\\ &= \sum_p (-1)^p \sum_i \int_X e^{(p)}(x,x,t) d\mathrm{Vol}(x).\end{aligned}$$

Proposition 5.5.1. $e(x,x,t) \sim (4\pi t)^{-n/2} \sum_{k=0}^\infty u_k(x,t) t^k$, where $u_k(x,t)$ is explicitly given in terms of components of curvatures.

Hence, as $t \longrightarrow 0$, we have

$$\mathcal{X} \sim \frac{1}{4\pi t}^{n/2} \sum_{k=0}^{\infty} \left(\int_X \sum_{p=0}^{\infty} (-1)^p \mathrm{tr} u_k^p(x,x) d\mathrm{Vol}(x) \right) t^k.$$

This implies

$$4\pi^{n/2}\int_X \sum_{p=0}^{\infty}(-1)\mathrm{tr}u_k^p(x,x)d\mathrm{Vol}(x) = \begin{cases} 0 & \text{if } k \neq n/2, \\ \mathcal{X}(X) & k = n/2. \end{cases}$$

Theorem 5.5.1 (Gauss-Bonet)**.** Let $n = \dim(X)$ be even. Then

$$\mathcal{X}(X) = \int_X \omega,$$

where ω is given in a local frame by

$$\omega = c_n \sum_{\sigma,\tau}(\mathrm{sign}\sigma)(\mathrm{sign}\tau)R_{\sigma(1)\sigma(2)\tau(1)\tau(2)}\cdots R_{\sigma(n-1)\sigma(n)\tau(n-1)\tau(n)}$$

and $c_n = \frac{(-1)^{n/2}}{(8\pi)^{n/2}\left(\frac{n}{2}\right)!}$.

For $n = 2$, $\omega = \frac{1}{8\pi}(R_{1212} - R_{1221} - R_{2112} - R_{2121})dA = -\frac{1}{2\pi}R_{1212}^{dA} = \frac{K}{2\pi}dA$ where K is the Gaussian curvature.

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