Baer sums

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Abstract

We define Baer sums in the category of left R-modules. This operation makes $\mathrm{Ext}^1(A,B)$ into an abelian group, where $\mathrm{Ext}^1(A,B)$ is considered as a set of classes of short exact sequences under certain equivalence relation. After proving this fact, we extend the notion of Baer sum to the category of chain complexes $\mathrm{Ch}(R)$.

Contents

1 Baer sums in Mod_R

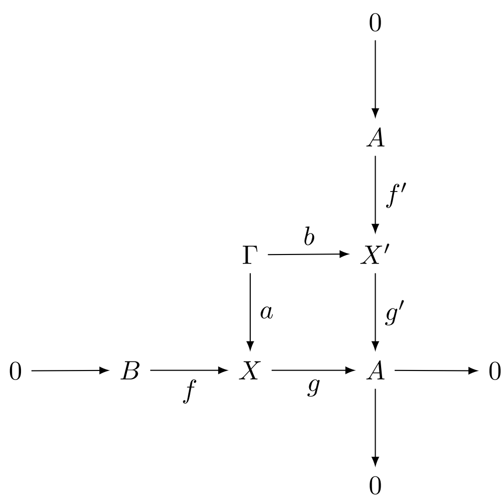
Let $0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow B \longrightarrow X' \longrightarrow A \longrightarrow 0$ be two short exact sequences in Mod_R. We shall say that they are equivalent if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \Big\downarrow = & & \Big\downarrow \varphi & & \Big\downarrow = & & \\ 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

We shall denote this fact $0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0 \cong 0 \longrightarrow B \longrightarrow X' \longrightarrow A \longrightarrow 0$. It is easy to see that $\cong$ is an equivalence relation on the set of short exact sequences. Denote $\mathrm{Ext}^1(A,B)$ the quotient set under this relation.

Now we define a way to "add" two classes in $\mathrm{Ext}^1(A,B)$. We are given two classes of short exact sequences $0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0$ and $0 \longrightarrow B \xrightarrow{f'} X' \xrightarrow{g'} A \longrightarrow 0$. Consider the pullback of g and g', which in Mod_R is given by $\Gamma = X \coprod_A X' := \{(x,x') \in X \oplus X' : g(x) = g'(x')\}$.

We have a commutative diagram



where the maps a and b are given by $a(x, x') = x$ and $b(x, x') = x'$. The set $\Delta = \{(f(b), -f'(b)) : b \in B\}$ is a submodule of Γ. Then we can take the quotient module $Y = \Gamma/\Delta$. Consider the maps $\overline{f} : B \longrightarrow Y$ and $\overline{g} : Y \longrightarrow A$ given by $\overline{f}(b) = [f(b), 0] = [0, f'(b)]$ and $\overline{g}([x, x']) = g(x) = g'(x')$. If $[x, x'] = [y, y']$ then $(x, x') - (y, y') = (f(b), -f'(b))$. We have $g(x) = g(y + f(b)) = g(y) + g \circ f(b) = g(y) + 0 = g(y)$. So $\overline{g}$ is well defined.

Proposition 1.1. The sequence $0 \longrightarrow B \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} A \longrightarrow 0$ is exact. This sequence is known as the **Baer sum** of $0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow B \longrightarrow X' \longrightarrow A \longrightarrow 0$.

Proof:

- $\overline{f}$ is injective: Suppose $[f(b), 0] = [0, 0]$. Then there exists $b' \in B$ such that $f(b) = f(b')$ and $0 = -f'(b')$. Since f' is injective, we get $b' = 0$ and $f(b) = 0$. Since f in injective, we get $b = 0$.
- $\overline{g}$ is surjective: Let $a \in A$. Since g and g' are surjective, there exist $x \in X$ and $x' \in X'$ such that $a = g(x) = g'(x')$. So $(x, x') \in \Gamma$ and $a = \overline{g}([x, x'])$.
- $\mathrm{Im}(\overline{f}) \subseteq \mathrm{Ker}(\overline{g})$: Let $b \in B$. We have $\overline{g} \circ \overline{f}(b) = \overline{g}([f(b), 0]) = g(f(b)) = 0$.
- $\mathrm{Im}(\overline{f}) \supseteq \mathrm{Ker}(\overline{g})$: Let $[x, x'] \in \mathrm{Ker}(\overline{g})$. We have $g(x) = g'(x') = 0$. Then there exist unique $b, b' \in B$ such that $x = f(b)$ and $x' = f'(b')$. We have

$$x' = f'(b') = -f'(-b'),$$
$$x = f(b) = f(b) + f(b') - f(b') = f(b + b') + f(-b').$$

  It follows $[x, x'] = [f(b + b'), 0] = \overline{f}(b + b')$.

□

The previous proposition gives rise to a binary operation $+_B : \mathrm{Ext}^1(A, B) \times \mathrm{Ext}^1(A, B) \longrightarrow \mathrm{Ext}^1(A, B)$.

Proposition 1.2. The Baer sum $+_B$ is a well defined binary operation that makes $\text{Ext}^1(A, B)$ into an abelian group.

Proof:

- $+_B$ is well defined: Suppose

$$(0 \longrightarrow B \xrightarrow{f} X_1 \xrightarrow{g} A \longrightarrow 0) \cong (0 \longrightarrow B \xrightarrow{f'} X_1' \xrightarrow{g'} A \longrightarrow 0)$$
$$(0 \longrightarrow B \xrightarrow{s} X_2 \xrightarrow{t} A \longrightarrow 0) \cong (0 \longrightarrow B \xrightarrow{s'} X_2' \xrightarrow{t'} A \longrightarrow 0)$$

Consider the modules

$$\begin{aligned} \Gamma &= \{(x_1, x_2) \in X_1 \oplus X_2 : g(x_1) = t(x_2)\}, \\ \Gamma' &= \{(x_1', x_2') \in X_1' \oplus X_2' : g'(x_1') = t'(x_2')\}, \\ \Delta &= \{(f(b), -s(b)) : b \in B\}, \\ \Delta' &= \{(f'(b), -s'(b)) : b \in B\}. \end{aligned}$$

We know there exist commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{f} & X_1 & \xrightarrow{g} & A & \longrightarrow & 0 \\ & & \Big\downarrow = & & \Big\downarrow \varphi_1 & & \Big\downarrow = & & \\ 0 & \longrightarrow & B & \xrightarrow[f']{} & X_1' & \xrightarrow[g']{} & A & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{s} & X_2 & \xrightarrow{t} & A & \longrightarrow & 0 \\ & & \Big\downarrow = & & \Big\downarrow \varphi_2 & & \Big\downarrow = & & \\ 0 & \longrightarrow & B & \xrightarrow[s']{} & X_2' & \xrightarrow[t']{} & A & \longrightarrow & 0 \end{array}$$

Define a map $\varphi : Y \longrightarrow Y'$ by $\varphi([x_1, x_2]) = [\varphi_1(x_1), \varphi_2(x_2)]$. This map is well defined. For if $[x_1, x_2] = [x_1', x_2']$ then $x_1 - x_1' = f(b)$ and $x_2 - x_2' = -s(b)$. So we get

$$\begin{aligned} \varphi_1(x_1) - \varphi_1(x_1') &= \varphi_1(f(b)) = f'(b); \\ \varphi_2(x_2) - \varphi_2(x_2') &= -\varphi_2(s(b)) = -s'(b). \end{aligned}$$

It follows $[\varphi_1(x_1), \varphi_2(x_2)] = [\varphi_1(x_1), \varphi_2(x_2)]$. It is clear that φ is a homomorphism of left R-modules. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\overline{f}} & Y & \xrightarrow{\overline{g}} & A & \longrightarrow & 0 \\ & & \Big\downarrow = & & \Big\downarrow \varphi & & \Big\downarrow = & & \\ 0 & \longrightarrow & B & \xrightarrow[\overline{f'}]{} & Y' & \xrightarrow[\overline{g'}]{} & A & \longrightarrow & 0 \end{array}$$

commutes since

$$\varphi \circ \overline{f}(b) = \varphi([f(b), 0]) = [\varphi_1(f(b)), \varphi_2(0)] = [f'(b), 0] = \overline{f'}(b),$$
$$\overline{g'} \circ \varphi([x_1, x_2]) = \overline{g'}([\varphi_1(x_1), \varphi_2(x_2)]) = g'(\varphi_1(x_1)) = g(x_1) = \overline{g}([x_1, x_2]).$$

Hence

$$(0 \longrightarrow B \xrightarrow{f} X_1 \xrightarrow{g} A \longrightarrow 0) +_B (0 \longrightarrow B \xrightarrow{s} X_2 \xrightarrow{t} A \longrightarrow 0)$$
$$\cong$$
$$(0 \longrightarrow B \xrightarrow{f'} X_1' \xrightarrow{g'} A \longrightarrow 0) +_B (0 \longrightarrow B \xrightarrow{s'} X_2' \xrightarrow{t'} A \longrightarrow 0)$$

- $+_B$ is commutative: Consider two classes of exact sequences $0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0$ and $0 \longrightarrow B \xrightarrow{f'} X' \xrightarrow{g'} A \longrightarrow 0$. Denote

$$\Gamma = \{(x, x') \in X \oplus X' : g(x) = g'(x')\},$$
$$\Gamma' = \{(x', x) \in X \oplus X' : g'(x') = g(x)\},$$
$$\Delta = \{(f(b), -f'(b)) : b \in B\},$$
$$\Delta' = \{(f'(b), -f(b)) : b \in B\}.$$

Consider the map $\varphi : Y \longrightarrow Y'$. It is clear that φ is a well defined homomorphism of left R-modules. Also,

$$\varphi \circ \overline{f}(b) = \varphi([f(b), 0]) = [0, f(b)] = [f'(b), 0] = \overline{f'}(b),$$
$$\overline{g'} \circ \varphi([x, x']) = \overline{g'}([x', x]) = g'(x') = g(x) = \overline{g}([x, x']).$$

So the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\overline{f}} & Y & \xrightarrow{\overline{g}} & A & \longrightarrow & 0 \\
 & & \big\downarrow = & & \big\downarrow \varphi & & \big\downarrow = & & \\
0 & \longrightarrow & B & \xrightarrow[\overline{f'}]{} & Y' & \xrightarrow[\overline{g'}]{} & A & \longrightarrow & 0
\end{array}$$

commutes and hence $+_B$ is commutative.

- $+_B$ is associative: Consider the following classes of sequences

$$0 \longrightarrow B \xrightarrow{f_1} X_1 \xrightarrow{g_1} A \longrightarrow 0,$$
$$0 \longrightarrow B \xrightarrow{f_2} X_2 \xrightarrow{g_2} A \longrightarrow 0,$$
$$0 \longrightarrow B \xrightarrow{f_3} X_3 \xrightarrow{g_3} A \longrightarrow 0.$$

We use the following notation:

$$
\begin{aligned}
\Gamma_{12} &= \{(x_1, x_2) \in X_1 \oplus X_2 : g_1(x_1) = g_2(x_2)\}, \\
\Delta_{12} &= \{(f_1(b), -f_2(b)) : b \in B\}, \\
Y_{12} &= \Gamma_{12}/\Delta_{12}, \\
\Gamma_{12,3} &= \{([x_1, x_2], x_3) \in Y_{12} \oplus X_3 : g_{12}([x_1, x_2]) = g_3(x_3)\} \\
&= \{([x_1, x_2], x_3) \in Y_{12} \oplus X_3 : g_1(x_1) = g_2(x_2) = g_3(x_3)\}, \\
\Delta_{12,3} &= \{(f_{12}(b), -f_3(b)) : b \in B\}, \\
Y_{12,3} &= \Gamma_{12,3}/\Delta_{12,3}, \\
\Gamma_{23} &= \{(x_2, x_3) \in X_2 \oplus X_3 : g_2(x_2) = g_3(x_3)\}, \\
\Delta_{23} &= \{(f_2(b), -f_3(b)) : b \in B\}, \\
Y_{23} &= \Gamma_{23}/\Delta_{23}, \\
\Gamma_{1,23} &= \{(x_1, [x_2, x_3]) \in X_1 \oplus Y_{23} : g_1(x_1) = g_{23}([x_2, x_3])\} \\
&= \{(x_1, [x_2, x_3]) \in X_1 \oplus Y_{23} : g_1(x_1) = g_2(x_2) = g_3(x_3)\}, \\
\Delta_{1,23} &= \{(f_1(b), -f_{23}(b)) : b \in B\}, \\
Y_{1,23} &= \Gamma_{1,23}/\Delta_{1,23}.
\end{aligned}
$$

Let $\varphi : Y_{12,3} \longrightarrow Y_{1,23}$ be the map given by $\varphi(\langle [x_1, x_2], x_3\rangle) = \langle x_1, [x_2, x_3]\rangle$. We check this map is well defined. Suppose $\langle [x_1, x_2], x_3\rangle = \langle [x_1', x_2'], x_3'\rangle$. Then there exists $b \in B$ such that

$$
\begin{aligned}
[x_1, x_2] - [x_1', x_2'] &= f_{12}(b), \\
x_3 - x_3' &= -f_3(b).
\end{aligned}
$$

It follows from the first equality that there exists $b' \in B$ such that

$$
\begin{aligned}
x_1 - x_1' &= f_1(b) + f_1(b'), \\
x_2 - x_2' &= -f_2(b')
\end{aligned}
$$

So we get

$$
\begin{aligned}
x_1 - x_1' &= f_1(b + b'), \\
x_2 - x_2' &= -f_2(b') = -f_2(b + b') + f_2(b), \\
x_3 - x_3' &= -f_3(b).
\end{aligned}
$$

Hence, $\langle x_1, [x_2, x_3]\rangle = \langle x_1', [x_2', x_3']\rangle$. It is clear that φ is a homomorphism of left R-modules. Moreover, the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{f_{12,3}} & Y_{12,3} & \xrightarrow{g_{12,3}} & A & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow \varphi & & \downarrow = & & \\
0 & \longrightarrow & B & \xrightarrow[f_{1,23}]{} & Y_{1,23} & \xrightarrow[g_{1,23}]{} & A & \longrightarrow & 0
\end{array}
$$

commutes since

$$
\varphi \circ f_{12,3}(b) = \varphi(\langle f_{12}(b), 0\rangle) = \varphi(\langle [f_1(b), 0], 0\rangle) = \langle f_1(b), [0, 0]\rangle = f_{1,23}(b),
$$
$$
g_{1,23} \circ \varphi(\langle [x_1, x_2], x_3\rangle) = g_{1,23}(\langle x_1, [x_2, x_3]\rangle) = g_1(x_1) = g_{12}([x_1, x_2]) = g_{12,3}(\langle [x_1, x_2], x_3\rangle).
$$

- The class of $0 \longrightarrow B \xrightarrow{i_B} A \oplus B \xrightarrow{p_A} A \longrightarrow 0$ is the zero element of $[\mathrm{Ext}^1(A,B), +_B]$: We want to show that

$$(0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0) +_B (0 \longrightarrow B \xrightarrow{i_B} A \oplus B \xrightarrow{p_A} A \longrightarrow 0)$$
$$\cong$$
$$(0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0)$$

In this case, we have

$$\begin{aligned} \Gamma &= \{(x,a,b) \in X \oplus (A \oplus B) : g(x) = a\}, \\ \Delta &= \{(f(b),0,b) : b \in B\}, \\ Y &= \Gamma/\Delta. \end{aligned}$$

Consider the homomorphism of left R-modules $\varphi : X \longrightarrow Y$ given by $\varphi(x) = [x, g(x), 0]$. We have

$$\begin{aligned} \varphi \circ f(b) &= [f(b),0,0] = [0,0,b] = \overline{i_B}(b), \\ \overline{p_A} \circ \varphi(x) &= \overline{p_A}([x,g(x),0]) = g(x). \end{aligned}$$

So the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{f} & X & \xrightarrow{g} & A & \longrightarrow & 0 \\ & & \big\downarrow = & & \big\downarrow \varphi & & \big\downarrow = & & \\ 0 & \longrightarrow & B & \xrightarrow[\overline{i_B}]{} & Y & \xrightarrow[\overline{p_A}]{} & A & \longrightarrow & 0 \end{array}$$

and the result follows.

- Every class in $\mathrm{Ext}^1(A,B)$ has an inverse element with respect to $+_B$: Consider a class of a sequence $0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0$, and the class of $0 \longrightarrow B \xrightarrow{f} X \xrightarrow{-g} A \longrightarrow 0$. We add the previous two classes. We have

$$\begin{aligned} \Gamma &= \{(x,x') \in X \oplus X : g(x) = -g(x')\}, \\ \Delta &= \{(f(b),-f(b)) : b \in B\}, \\ Y &= \Gamma/\Delta. \end{aligned}$$

Let $[x,x'] \in Y$. Then $g(x+x') = 0$. Since the first sequence is exact and f is a monomorphism, there exists a unique $b \in B$ such that $x + x' = f(b)$. Define $\varphi : Y \longrightarrow A \oplus B$ by $\varphi([x,x']) = (g(x), b)$. It is easy to check that φ is a well defined homomorphism of left R-modules and that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\overline{f}} & Y & \xrightarrow{\overline{g}} & A & \longrightarrow & 0 \\ & & \big\downarrow = & & \big\downarrow \varphi & & \big\downarrow = & & \\ 0 & \longrightarrow & B & \xrightarrow[i_B]{} & A \oplus B & \xrightarrow[p_A]{} & A & \longrightarrow & 0 \end{array}$$

Hence the result follows.

$\square$

2 Baer sums in $\mathrm{Ch}(R)$

Consider two exact sequences $0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0$ and $0 \longrightarrow B \xrightarrow{f'} X' \xrightarrow{g'} A \longrightarrow 0$ in $\mathrm{Ch}(R)$. We shall say that they are equivalent if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{f} & X & \xrightarrow{g} & A & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow{\varphi} & & \big\downarrow{=} & & \\ 0 & \longrightarrow & B & \xrightarrow[f']{} & X' & \xrightarrow[g']{} & A & \longrightarrow & 0 \end{array}$$

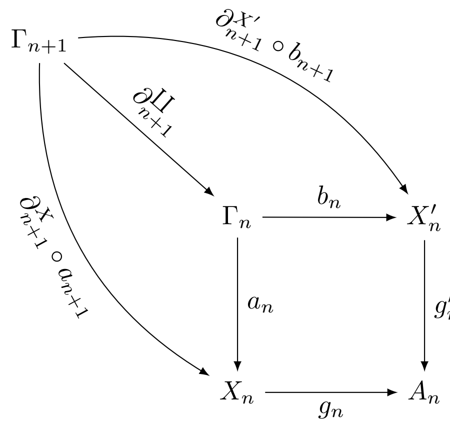
It follows that for each $n \in \mathbb{Z}$ we have a commutative diagram in Mod_R

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \xrightarrow{f_n} & X_n & \xrightarrow{g_n} & A_n & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow{\varphi_n} & & \big\downarrow{=} & & \\ 0 & \longrightarrow & B_n & \xrightarrow[f'_n]{} & X'_n & \xrightarrow[g'_n]{} & A_n & \longrightarrow & 0 \end{array}$$

Consider the pullback diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \big\downarrow & & \\ & & & & & & A_n & & \\ & & & & & & \big\downarrow{f'_n} & & \\ & & & & \Gamma_n & \xrightarrow{b_n} & X'_n & & \\ & & & & \big\downarrow{a_n} & & \big\downarrow{g'_n} & & \\ 0 & \longrightarrow & B_n & \xrightarrow[f_n]{} & X_n & \xrightarrow[g_n]{} & A_n & \longrightarrow & 0 \\ & & & & & & \big\downarrow & & \\ & & & & & & 0 & & \end{array}$$

where $\Gamma_n = X_n \coprod_{A_n} X'_n$. By the universal property of pullbacks, there exists a unique map $\partial^{\coprod}_{n+1} : \Gamma_{n+1} \longrightarrow \Gamma_n$ such that the following diagram commutes:



Using a similar argument, one can show that $X \coprod_A X' = (X_n \coprod_{A_n} X'_n, \partial_n^{\coprod})_{n\in\mathbb{Z}}$ is a chain complex. It follows that $a : X \coprod_A X' \longrightarrow X := (a_n)_{n\in\mathbb{Z}}$ and $b : X \coprod_A X' \longrightarrow X' := (b_n)_{n\in\mathbb{Z}}$ are chain transformations. It is easy to check that $(X \coprod_A X, a, b)$ is the pullback of g and g'. We denote $\Gamma = X \coprod_A X'$. For each $n \in \mathbb{Z}$, let $\Delta_n = \{(f_n(b_n), -f'_n(b_n)) : b_n \in B_n\}$ and $Y_n = \Gamma_n/\Delta_n$. Define maps $\partial_n^Y : Y_n \longrightarrow Y_{n-1}$ by $\partial_n^Y([x_n, x'_n]) = [\partial_n^{\coprod}(x_n, x'_n)]$.

Proposition 2.1. $Y := (Y_n, \partial_n^Y)_{n\in\mathbb{Z}}$ defines a chain complex.

Proof: We first check that each ∂_n^Y is well defined. Notice that each map $\partial_n^{\coprod}$ has two components, namely $a_{n-1} \circ \partial_n^{\coprod}$ and $b_{n-1} \circ \partial_n^{\coprod}$. Suppose $[x_n, x'_n] = [y_n, y'_n]$. Then there exists $b_n \in B_n$ such that

$$\begin{aligned} x_n - x'_n &= f_n(b_n), \\ y_n - y'_n &= -f'_n(b_n). \end{aligned}$$

We have

$$\begin{aligned} (x_n, x'_n) &= (y_n, y'_n) + (f_n(b_n), -f'_n(b_n)), & (1)\\ \partial_n^{\coprod}(x_n, x'_n) &= \partial_n^{\coprod}(y_n, y'_n) + \partial_n^{\coprod}(f_n(b_n), -f'_n(b_n)), & (2)\\ a_{n-1} \circ \partial_n^{\coprod}(x_n, x'_n) &= a_{n-1} \circ \partial_n^{\coprod}(y_n, y'_n) + a_{n-1} \circ \partial_n^{\coprod}(f_n(b_n), -f'_n(b_n)), & (3)\\ b_{n-1} \circ \partial_n^{\coprod}(x_n, x'_n) &= b_{n-1} \circ \partial_n^{\coprod}(y_n, y'_n) + b_{n-1} \circ \partial_n^{\coprod}(f_n(b_n), -f'_n(b_n)). & (4) \end{aligned}$$

By (1) and the fact that a is a chain transformation, we have

$$\begin{aligned} a_{n-1} \circ \partial_n^{\coprod}(x_n, x'_n) &= a_{n-1} \circ \partial_n^{\coprod}(y_n, y'_n) + \partial_n^X \circ a_n(f_n(b_n), -f'_n(b_n)) \\ &= a_{n-1} \circ \partial_n^{\coprod}(y_n, y'_n) + \partial_n^X \circ f_n(b_n) \\ &= a_{n-1} \circ \partial_n^{\coprod}(y_n, y'_n) + f_{n-1} \circ \partial_n^B(b_n) \end{aligned}$$

Similarly $b_{n-1} \circ \partial_n^{\coprod}(x_n, x'_n) = b_{n-1} \circ \partial_n^{\coprod}(y_n, y'_n) - f'_{n-1} \circ \partial_n^B(b_n)$. Hence ∂_n^Y is well defined. It is easy to check that $\partial_{n-1}^Y \circ \partial_n^Y = 0$. □

By the first section, we have exact sequences $0 \longrightarrow B_n \xrightarrow{\overline{f_n}} Y_n \xrightarrow{\overline{g_n}} A_n \longrightarrow 0$ where

$$\begin{aligned} \overline{f_n}(b_n) &= [f_n(b_n), 0] = [0, f'_n(b_n)], \\ \overline{g_n}([x_n, x'_n]) &= g_n(x_n) = g'_n(x'_n). \end{aligned}$$

Proposition 2.2. $\overline{f} := (\overline{f_n})_{n\in\mathbb{Z}}$ and $\overline{g} := (\overline{g_n})_{n\in\mathbb{Z}}$ are chain transformations.

Proof: We check that the following squares commute:

$$\begin{array}{ccc} B_n & \xrightarrow{\partial_n^B} & B_{n-1} \\ \downarrow \overline{f_n} & & \downarrow \overline{f_{n-1}} \\ Y_n & \xrightarrow[\partial_n^Y]{} & Y_{n-1} \end{array} \qquad \begin{array}{ccc} Y_n & \xrightarrow{\partial_n^Y} & Y_{n-1} \\ \downarrow \overline{g_n} & & \downarrow \overline{g_{n-1}} \\ A_n & \xrightarrow[\partial_n^A]{} & A_{n-1} \end{array}$$

We have

$$\begin{aligned} \partial_n^Y \circ \overline{f_n}(b_n) &= \partial_n^Y([f_n(b_n), 0]) = [\partial_n^{\coprod}(f_n(b_n), 0)] \\ &= [(a_{n-1} \circ \partial_n^{\coprod}(f_n(b_n), 0), b_{n-1} \circ \partial_n^{\coprod}(f_n(b_n), 0))] \\ &= [(\partial_n^X \circ a_n(f_n(b_n), 0), \partial_n^X \circ b_n(f_n(b), 0))] \\ &= [(\partial_n^X \circ f_n(b_n), \partial_n^X(0))] \\ &= [(f_{n-1}(\partial_n^B(b_n)), 0)] \\ &= \overline{f_{n-1}} \circ \partial_n^B(b_n), \\ \overline{g_{n-1}} \circ \partial_n^Y([x_n, x'_n]) &= \overline{g_{n-1}}([\partial_n^{\coprod}(x_n, x'_n)]) = g_{n-1}(a_{n-1} \circ \partial_n^{\coprod}(x_n, x'_n)) \\ &= g_{n-1} \circ a_{n-1} \circ \partial_n^{\coprod}(x_n, x'_n) \\ &= g_{n-1} \circ \partial_n^X \circ a_n(x_n, x'_n) \\ &= \partial_n^A \circ g_n(x_n) \\ &= \partial_n^A \circ \overline{g_n}([x_n, x'_n]). \end{aligned}$$

□

Therefore, we have a short exact sequence in of complexes $0 \longrightarrow B \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} A \longrightarrow 0$. This sequence is called the **Baer sum** of $0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0$ and $0 \longrightarrow B \xrightarrow{f'} X' \xrightarrow{g'} A \longrightarrow 0$. Denote this operation by $+_B$. As we did in the previous section, one can show that $(\mathrm{Ext}^1(A, B), +_B)$ is an abelian group.

References

[1] Passman, D. S. *A Course in Ring Theory*. American Mathematical Society. (2004).