Exact model structures from Auslander-Buchweitz Approximation Theory

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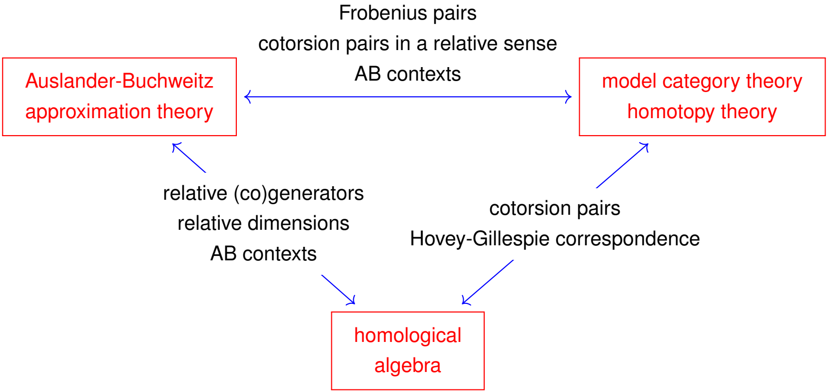
motivation


Frobenius pairs
cotorsion pairs in a relative sense
AB contexts
Auslander-Buchweitz
approximation theory
model category theory
homotopy theory
relative (co)generators
relative dimensions
AB contexts
cotorsion pairs
Hovey-Gillespie correspondence
homological
algebra

outline

1. Model structures on exact categories (Hovey-Gillespie correspondence)
2. Cotorsion pairs relative to thick subcategories
3. Frobenius pairs in abelian categories
4. Auslander-Buchweitz model structures
5. Correspondences to AB contexts

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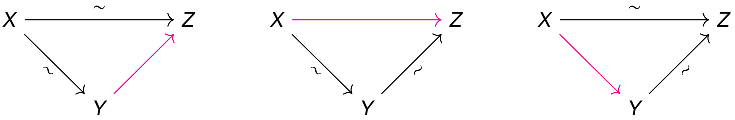
what is a model structure?

Let $\mathcal{C}$ be a bicomplete category. A **model structure** on $\mathcal{C}$ (in the sense of Hovey-Tierney) is a triple $(\mathfrak{C}_{\rm of}, \mathfrak{F}_{\rm ib}, \mathfrak{W}_{\rm eak})$ of classes of morphisms in $\mathcal{C}$, called **cofibrations**, **fibrations** and **weak equivalences**, respectively, such that:

m1 $\mathfrak{W}_{\rm eak}$ satisfies the 3 × 2 property:



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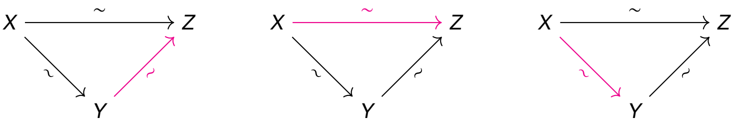

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m2 $\mathfrak{C}_{\text{of}}$ (resp., $\mathfrak{C}_{\text{of}} \cap \mathfrak{W}_{\text{eak}}$) has the left lifting property with respect to $\mathfrak{F}_{\text{ib}} \cap \mathfrak{W}_{\text{eak}}$ (resp., $\mathfrak{F}_{\text{ib}}$).

$$\begin{array}{ccccc} & X & \longrightarrow & Z & \\ \mathfrak{C}_{\text{of}} \ni & \downarrow & & \downarrow \wr & \in \mathfrak{F}_{\text{ib}} \cap \mathfrak{W}_{\text{eak}} \\ & Y & \longrightarrow & W & \end{array} \qquad \begin{array}{ccccc} & X' & \longrightarrow & Z' & \\ \mathfrak{C}_{\text{of}} \cap \mathfrak{W}_{\text{eak}} \ni & \downarrow \wr & & \downarrow & \in \mathfrak{F}_{\text{ib}} \\ & Y' & \longrightarrow & W' & \end{array}$$

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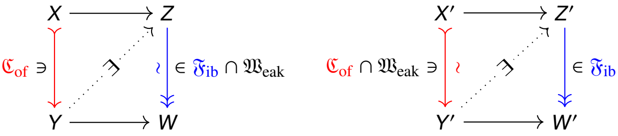

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m3 Every morphism in $\mathcal{C}$ can be factored (functorially) as a cofibration follows by a trivial fibration, and as a trivial cofibration followed by a fibration.

$$\begin{array}{c} X \\ \Big\downarrow f \\ Y \end{array}$$

Let C be a bicomplete category. A **model structure** on C (in the sense of Hovey-Tierney) is a triple $(\mathfrak{C}_{\text{of}}, \mathfrak{F}_{\text{ib}}, \mathfrak{W}_{\text{eak}})$ of classes of morphisms in C, called **cofibrations**, **fibrations** and **weak equivalences**, respectively, such that:

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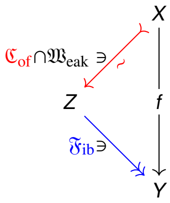

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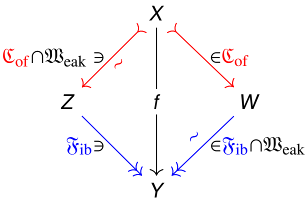

Given a model structure $(\mathfrak{C}_{\text{of}}, \mathfrak{F}_{\text{ib}}, \mathfrak{W}_{\text{eak}})$, we denote by:

- $\mathcal{Q}$ the class of **cofibrant objects**, *i.e.,* those objects X such that the only morphism

  $$I \rightarrowtail X$$

  is a **cofibration**.

- $\mathcal{R}$ the class of **fibrant objects**, *i.e.,* those objects Y such that the only morphism

  $$Y \twoheadrightarrow T$$

  is a **fibration**.

- $\mathcal{T}$ the class of **trivial objects**, *i.e.,* those objects X such that the only morphism

  $$I \rightarrowtail X$$

  is a **weak equivalence**.

Let C be a model category equipped with a model structure $\mathcal{M} = (\mathfrak{C}_{\text{of}}, \mathfrak{F}_{\text{ib}}, \mathfrak{W}_{\text{eak}})$.

The **homotopy category** $\mathbf{Ho}(C)$ of C (with respect to $\mathcal{M}$) can be defined:

i. Via localization of C at the class $\mathfrak{W}_{\text{eak}}$ of weak equivalences:

$$\mathbf{Ho}(C) := C[\mathfrak{W}_{\text{eak}}^{-1}].$$

ii. Via homotopy relations.

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The **homotopy category** **Ho**$(\mathcal{C})$ of $\mathcal{C}$ (with respect to $\mathcal{M}$) can be defined:

i Via localization of $\mathcal{C}$ at the class $\mathfrak{W}_{\text{eak}}$ of weak equivalences:

$$\mathbf{Ho}(\mathcal{C}) := \mathcal{C}[\mathfrak{W}_{\text{eak}}^{-1}].$$

ii Via homotopy relations. ✓

when are two morphisms left homotopic?

Let $X, Y \in \text{Ob}(\mathcal{C})$ and $f, g \colon X \to Y$ be two morphisms in $\mathcal{C}$.

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$$X \xhookrightarrow{\mu_0} X \amalg X \xhookleftarrow{\mu_1} X$$

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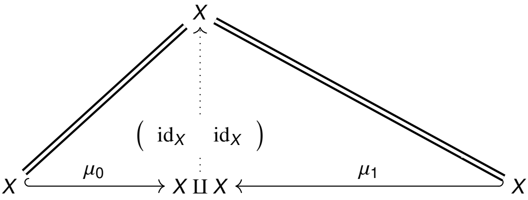

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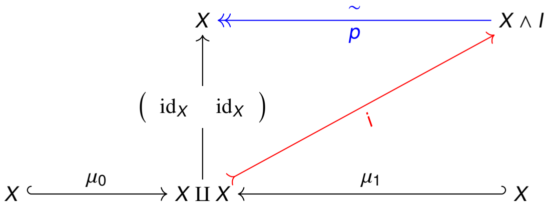

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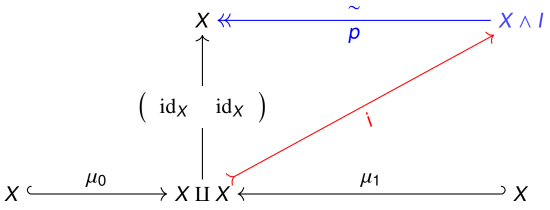

cylinder object

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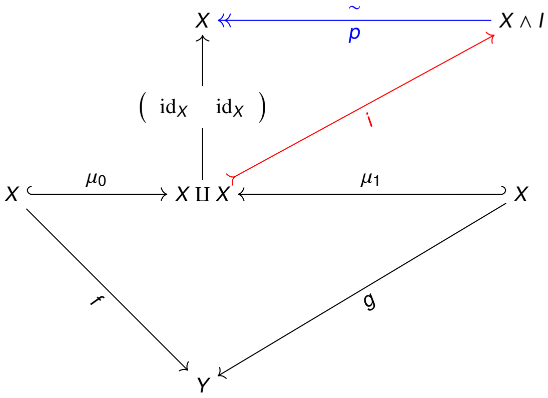

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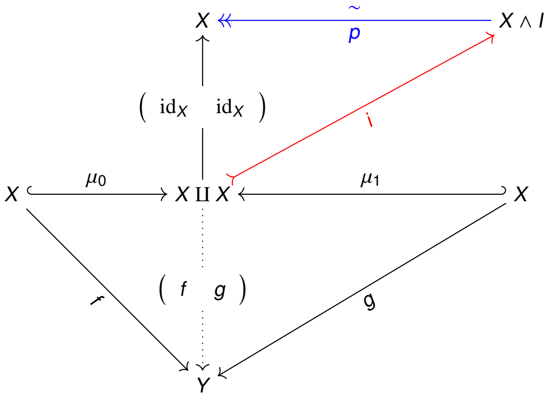

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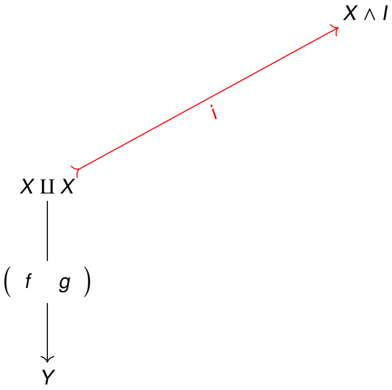

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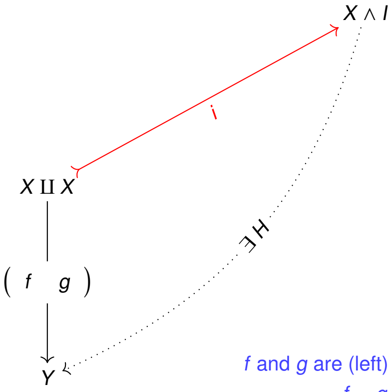

f and g are (left) homotopic

$f \sim g$

definition of homotopy categories

Let $X, Y \in \mathrm{Ob}(\mathcal{C})$.

$$X \xrightarrow[f]{} Y$$

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$$1 \longrightarrow X \xrightarrow[f]{} Y$$

definition of homotopy categories

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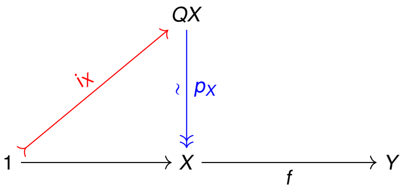

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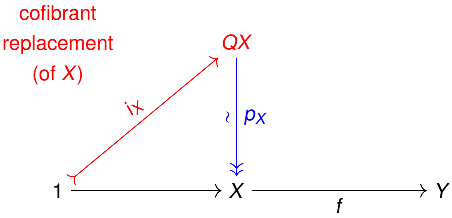

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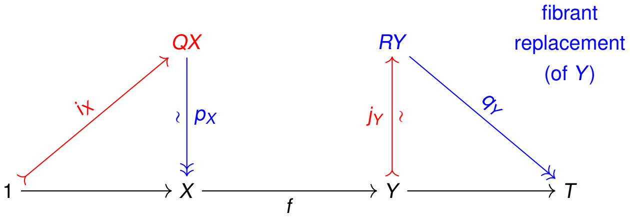

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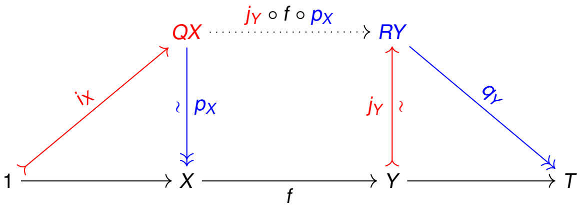

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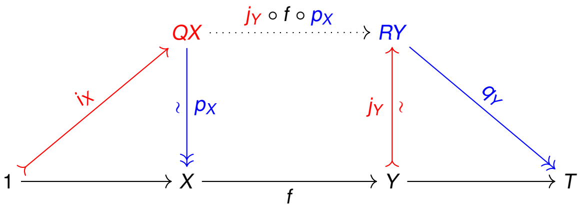


Theorem (W.G. Dwyer, P. S. Hirschhorn, D. M. Kan)

For every $X, Y \in \mathrm{Ob}(\mathcal{C})$, the mapping $f \mapsto j_Y \circ f \circ p_X$ gives rise to a natural isomorphism:

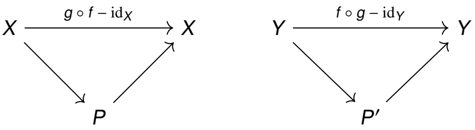
$$\mathrm{Hom}_{\boldsymbol{Ho}(\mathcal{C})}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(QX, RY)/\sim$$

examples

- the category of modules over a quasi-Frobenius ring:

Let R be a quasi-Frobenius ring (*i.e.,* left noetherian and self injective), and consider $\mathrm{Mod}(R)$.

Recall that an R-homomorphism $f\colon M \to N$ is a **stable equivalence** if there exists an R-homomorphism $g\colon N \to M$ such that:



$$\mathfrak{C}_{\mathrm{of}} := \text{monomorphisms}$$

$$\mathfrak{F}_{\mathrm{ib}} := \text{epimorphisms}$$

$$\mathfrak{W}_{\mathrm{eak}} := \text{stable equivalence}$$

$$\mathbf{Ho}(\mathrm{Ch}(R)) = \mathrm{Stmod}(R) := \underline{\mathrm{Mod}(R)}$$

Let $(\mathcal{E}, \tau)$ be an exact category. A model structure $\mathcal{M} = (\mathfrak{C}_{\text{of}}, \mathfrak{F}_{\text{ib}}, \mathfrak{W}_{\text{eak}})$ on $\mathcal{E}$ is called **exact** if the following conditions hold:

em1 $f \in \mathfrak{C}_{\text{of}}$ if, and only if, f is an admissible monomorphism with cokernel in $\mathcal{Q}$.

em2 $g \in \mathfrak{F}_{\text{ib}}$ if, and only if, g is an admissible epimorphism with kernel in $\mathcal{R}$.

A **cotorsion pair** in an exact category $(\mathcal{E}, \tau)$ is a pair $(\mathcal{F}, \mathcal{G})$ of classes of objects of $\mathcal{E}$ such that:

$$\mathcal{F} = {}^{\perp_{1,\tau}}\mathcal{G} := \{X \in \mathrm{Ob}(\mathcal{E}) : \mathrm{Ext}^1_\tau(X, G) = 0 \ \forall\ G \in \mathcal{G}\},$$
$$\mathcal{G} = \mathcal{F}^{\perp_{1,\tau}} := \{Y \in \mathrm{Ob}(\mathcal{E}) : \mathrm{Ext}^1_\tau(F, Y) = 0 \ \forall\ F \in \mathcal{F}\}.$$

A cotorsion pair $(\mathcal{F}, \mathcal{G})$ is called **complete** if for every object $X \in \mathrm{Ob}(\mathcal{E})$, there exist short exact sequences (in τ)

$$0 \to G \to F \to X \to 0,$$
$$0 \to X \to G' \to F' \to 0,$$

such that $F, F' \in \mathcal{F}$ and $G, G' \in \mathcal{G}$.

Let $(\mathcal{E}, \tau)$ be an exact category. A full subcategory $\mathcal{X} \subseteq \mathcal{E}$ is called **left thick** if:

1. $\mathcal{X}$ is closed under direct summands.
2. $\mathcal{X}$ is closed under extensions (in τ).
3. $\mathcal{X}$ is closed under kernels of admissible epimorphisms.

Let $(\mathcal{E}, \tau)$ be an exact category. Three full subcategories $\mathcal{A}, \mathcal{B}, \mathcal{W} \subseteq \mathcal{E}$ form a **Hovey triple** $(\mathcal{A}, \mathcal{B}, \mathcal{W})$ if:

1. $\mathcal{W}$ is thick.
2. $(\mathcal{A} \cap \mathcal{W}, \mathcal{B})$ is a complete cotorsion pair in $\mathcal{E}$.
3. $(\mathcal{A}, \mathcal{B} \cap \mathcal{W})$ is a complete cotorsion pair in $\mathcal{E}$.

Hovey-Gillespie correspondence

Theorem (M. Hovey (00') and J. Gillespie (11'))

Let $(\mathcal{E}, \tau)$ be a ***weakly idempotent complete*** *exact category (i.e., every split monomorphism has a cokernel and every split epimorphism has a kernel). Then there is a one-to-one correspondence between the class $\mathfrak{M}_{(\mathcal{E},\tau)}$ of exact model structures and:*

$$\mathfrak{T}_{(\mathcal{E},\tau)} := \{(\mathcal{A}, \mathcal{B}, \mathcal{W}) \subseteq \mathcal{E}^3 : (\mathcal{A}, \mathcal{B}, \mathcal{W}) \text{ is a Hovey triple in } \mathcal{E}\}.$$

$$\mathfrak{M}_{(\mathcal{E},\tau)} \longleftrightarrow \mathfrak{T}_{(\mathcal{E},\tau)}$$

$$\mathcal{M} \longmapsto (\mathcal{Q}, \mathcal{R}, \mathcal{T})$$

$$\left(\begin{array}{c} X \xrightarrow{f} Y \\ \text{ad. monic with} \\ \mathrm{CoKer}(f) \in \mathcal{A} \end{array} , \begin{array}{c} W \xrightarrow{g} Z \\ \text{ad. epic with} \\ \mathrm{Ker}(g) \in \mathcal{B} \end{array} , \begin{array}{c} X \xrightarrow{f} Y \xrightarrow{g} Z \\ \text{where } f \text{ is ad. monic} \\ \text{with } \mathrm{CoKer}(f) \in \mathcal{A} \cap \mathcal{T} \\ \text{and } g \text{ is ad. epic} \\ \text{with } \mathrm{Ker}(g) \in \mathcal{R} \cap \mathcal{T} \end{array} \right) \longleftarrow\!\shortmid (\mathcal{A}, \mathcal{B}, \mathcal{T})$$

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$\mathcal{S}$-cotorsion pairs

Let $\mathcal{C}$ be an abelian category and $\mathcal{S} \subseteq \mathcal{C}$ be a thick subcategory. Two classes $\mathcal{F}$ and $\mathcal{G}$ of objects of $\mathcal{C}$ form as **$\mathcal{S}$-cotorsion pair** in $\mathcal{C}$ if $(\mathcal{F}, \mathcal{G})$ is a complete cotorsion pair in $\mathcal{S}$, or equivalently:

scp1 $\mathcal{F}, \mathcal{G} \subseteq \mathcal{S}$ and $\mathcal{F}$ is closed under direct summands in $\mathcal{C}$.

scp2 $\mathcal{F}, \mathcal{G} \subseteq \mathcal{S}$ and $\mathcal{G}$ is closed under direct summands in $\mathcal{C}$.

scp3 $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = 0$: this means $\mathrm{Ext}^1_{\mathcal{C}}(F, G) = 0$ for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

scp4 Every $S \in \mathcal{S}$ has a epic $\mathcal{F}$-precover with kernel in $\mathcal{G}$.

scp5 Every $S \in \mathcal{S}$ has a monic $\mathcal{G}$-preenvelope with cokernel in $\mathcal{F}$.

relative generators and cogenerators

Let $\mathcal{X}$ and ω be two classes of objects in an abelian category $\mathcal{C}$. Recall that ω is a **relative $\mathcal{X}$-injective cogenerator in $\mathcal{X}$** if:

1. $\omega \subseteq \mathcal{X}$.
2. For every $X \in \mathcal{X}$, there is a short exact sequence
$$0 \to X \to W \to X' \to 0.$$
with $W \in \omega$ and $X' \in \mathcal{X}$.
3. $\mathrm{id}_{\mathcal{X}}(\omega) = 0$.

Frobenius pairs

Two classes of objects $\mathcal{X}$ and ω in an abelian category $\mathcal{C}$ form a (**left**) **Frobenius pair** $(\mathcal{X}, \omega)$ in $\mathcal{C}$ if:

fp1 $\mathcal{X}$ is left thick, denoted $\mathcal{X} = \text{Thick}^-(\mathcal{X})$.

fp2 ω is closed under direct summands in $\mathcal{C}$.

fp3 ω is a relative $\mathcal{X}$-injective cogenerator in $\mathcal{X}$.

If in addition:

fp4 ω is a relative $\mathcal{X}$-projective generator in $\mathcal{X}$.

then we say that the Frobenius pair $(\mathcal{X}, \omega)$ is **strong**.

Example (strong Frobenius pairs)

$(\text{GProj}(R), \text{Proj}(R))$ is a strong Frobenius pair in $\text{Mod}(R)$.

relative cotorsion pairs from Frobenius pairs

Let $C \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{X}$ be a class of objects of $\mathcal{C}$. The **resolution dimension** of X is the smallest integer $n \geq 0$ such that there exists an exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to C \to 0$$

with $X_k \in \mathcal{X}$ for every $0 \leq k \leq n$.

Denote

$$\hat{\mathcal{X}} := \{C \in \mathrm{Ob}(\mathcal{C}) : \mathrm{resdim}_{\mathcal{X}}(C) < \infty\}.$$

Proposition

If $(\mathcal{X}, \omega)$ is a Frobenius pair in $\mathcal{C}$, then $(\mathcal{X}, \hat{\omega})$ is a $\hat{\mathcal{X}}$-cotorsion pair in $\mathcal{C}$ such that

$$\omega = \mathcal{X} \cap \hat{\omega}.$$

If in addition, $(\mathcal{X}, \omega)$ is strong, then $(\omega, \hat{\mathcal{X}})$ is a $\hat{\mathcal{X}}$-cotorsion pair in $\mathcal{C}$, and $\hat{\omega} = \mathrm{Thick}(\omega)$. Therefore, $(\mathcal{X}, \hat{\mathcal{X}}, \hat{\omega})$ is a Hovey triple in $\hat{\mathcal{X}}$.

Let $(\mathcal{X}, \omega)$ be a strong Frobenius pair in $\mathcal{C}$. We have two $\hat{\mathcal{X}}$-cotorsion pairs in $\mathcal{C}$, or equivalently:

two complete
cotorsion pairs in the exact category $\hat{\mathcal{X}}$

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from Frobenius pairs to model categories

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two complete cotorsion pairs in the exact category $\hat{\mathcal{X}}$ $(\mathcal{X}, \hat{\omega})$ and $(\omega, \hat{\mathcal{X}})$

$\downarrow$ $\|$ $\|$

two compatible and complete cotorsion pairs in the exact category $\hat{\mathcal{X}}$ $(\mathcal{X}, \hat{\omega} \cap \hat{\mathcal{X}})$ and $(\mathcal{X} \cap \hat{\omega}, \hat{\mathcal{X}})$

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↓

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↓

a Hovey triple

$(\mathcal{X}, \hat{\omega})$ and $(\omega, \hat{\mathcal{X}})$

$\|$ $\|$

$(\mathcal{X}, \hat{\omega} \cap \hat{\mathcal{X}})$ and $(\mathcal{X} \cap \hat{\omega}, \hat{\mathcal{X}})$

↘ ↙

$(\mathcal{X}, \hat{\mathcal{X}}, \hat{\omega})$

Theorem (Auslander-Buchweitz model structures)

Let $(\mathcal{X}, \omega)$ be a strong Frobenius pair in an abelian category $\mathcal{C}$. Then there exists a unique projective and exact model structure on the exact category $\hat{\mathcal{X}}$, called the ***projective Auslander-Buchweitz model structure*** *and denoted $\mathcal{M}^{\text{proj}}_{\text{AB}}(\mathcal{X}, \omega) := (\mathcal{X}, \hat{\mathcal{X}}, \hat{\omega})$, such that*

$$\mathcal{T} = \hat{\omega}, \ \ \mathcal{Q} = \mathcal{X} \ \text{ and } \ \mathcal{Q} \cap \mathcal{T} = \omega.$$

Proposition

Let $(\mathcal{X}, \omega)$ be a strong Frobenius pair in an abelian category $\mathcal{C}$, and $(\hat{\mathcal{X}}, \mathcal{M}^{\text{proj}}_{\text{AB}}(\mathcal{X}, \omega))$ its associated AB model category. Then for every $X, Y \in \text{Ob}(\hat{\mathcal{X}})$, there is a natural isomorphism

$$\text{Hom}_{\mathbf{Ho}(\hat{\mathcal{X}})}(X, Y) \cong \text{Hom}_{\hat{\mathcal{X}}}(QX, RY)/\sim \ \cong \text{Hom}_{\hat{\mathcal{X}}}(QX, Y)/\sim$$

where $f \sim g$ if, and only if, $f - g$ factors through an object in ω (*i.e.*, a projective object of $\hat{\mathcal{X}}$).

Example (applications to Gorenstein homological algebra)

Set $\mathcal{C} := \mathrm{Mod}(R)$, $\mathcal{X} := \mathrm{GProj}(R)$ and $\omega := \mathrm{Proj}(R)$.

We have an AB model category $(\mathrm{GProj}(R)^{<\infty}, \mathcal{M}^{\mathrm{proj}}_{\mathrm{AB}}(\mathrm{GProj}(R), \mathrm{Proj}(R)))$.

- If R is a quasi-Frobenius ring, then:
  - $\mathrm{Mod}(R) = \mathrm{GProj}(R)$.
  - $\mathcal{M}^{\mathrm{proj}}_{\mathrm{AB}}(\mathrm{GProj}(R), \mathrm{Proj}(R))$ is the Frobenius model structure on $\mathrm{Mod}(R)$.
  - $\mathbf{Ho}(\mathrm{Mod}(R)) = \mathrm{Stmod}(R)$.

outline

The following definition is motivated from

[M. Hashimoto, 2000 - Auslander-Buchweitz Approximations of Equivariant Modules]:

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of objects in an abelian category $\mathcal{C}$, and set $\omega := \mathcal{A} \cap \mathcal{B}$. We say that the pair $(\mathcal{A}, \mathcal{B})$ is a (**left**) **Auslander-Buchweitz precontext** in $\mathcal{C}$ if:

ab1 $(\mathcal{A}, \omega)$ is a (left) Frobenius pair.

ab2 $\mathcal{B}$ is right thick (denoted $\mathcal{B} = \text{Thick}^{+}(\mathcal{B})$).

If in addition, $(\mathcal{A}, \mathcal{B})$ satisfies:

ab3 $\mathcal{B} \subseteq \hat{\mathcal{A}}$,

we say that $(\mathcal{A}, \mathcal{B})$ is a (**left**) **AB context**.

AB contexts vs. relative cotorsion pairs

Theorem (from AB contexts to relative cotorsion pairs)

Let $(\mathcal{A}, \mathcal{B})$ be a left AB context and $\omega := \mathcal{A} \cap \mathcal{B}$. Then:

- $\omega = \mathcal{A} \cap \mathcal{A}^{\perp}$.
- $(\mathcal{A}, \mathcal{B})$ *is a* $\hat{\mathcal{A}}$*-cotorsion pair in* $\mathcal{C}$ *with* $\mathrm{id}_{\mathcal{A}}(\mathcal{B}) = 0$.

Theorem (from relative cotorsion pairs to AB contexts)

Let $(\mathcal{F}, \mathcal{G})$ be a $\mathrm{Thick}(\mathcal{F})$*-cotorsion pair in* $\mathcal{C}$ *with* $\mathrm{id}_{\mathcal{F}}(\mathcal{G}) = 0$*. Then:*

- $\mathrm{Thick}(\mathcal{F}) = \hat{\mathcal{F}}$.
- $(\mathcal{F}, \mathcal{G})$ *is a left AB-context in* $\mathcal{C}$.

correspondences

Theorem

For every abelian category $\mathcal{C}$, the following holds:

$$\mathfrak{F} := \{(\mathcal{X}, \omega) \subseteq \mathcal{C}^2 : (\mathcal{X}, \omega) \text{ is a Frobenius pair in } \mathcal{C}\} \qquad (\mathcal{X}, \omega) \quad (\mathcal{A}, \mathcal{A} \cap \mathcal{B})$$

$$\updownarrow \text{ one-to-one correspondence} \qquad\qquad \downarrow \qquad \uparrow$$

$$\mathfrak{C} := \{(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{C}^2 : (\mathcal{A}, \mathcal{B}) \text{ is an AB context in } \mathcal{C}\} \qquad (\mathcal{X}, \hat{\omega}) \quad (\mathcal{A}, \mathcal{B})$$

$$\|$$

$$\mathfrak{P} := \left\{ \begin{array}{c} (\mathcal{F}, \mathcal{G}) \subseteq \mathcal{C}^2 : (\mathcal{F}, \mathcal{G}) \text{ is a } \mathrm{Thick}(\mathcal{F})\text{-cotorsion} \\ \text{pair in } \mathcal{C} \text{ with } \mathrm{id}_{\mathcal{F}}(\mathcal{G}) = 0 \end{array} \right\}$$

Frobenius pairs vs. model categories

Consider the following subclass of $\mathfrak{F}$ and $\mathfrak{P}$, respectively:

$$\mathrm{s}\mathfrak{F} := \{(\mathcal{X},\omega) \subseteq \mathcal{C}^2 : (\mathcal{X},\omega) \text{ is a strong Frobenius pair in } \mathcal{C} \text{ such that } \mathrm{Proj}(\mathcal{C}) \subseteq \hat{\mathcal{X}}\},$$

$$\mathrm{s}\mathfrak{P} := \left\{ \begin{array}{c} (\mathcal{F},\mathcal{G}) \subseteq \mathcal{C}^2 : (\mathcal{F},\mathcal{G}) \text{ is a Thick}(\mathcal{F})\text{-cotorsion pair in } \mathcal{C} \\ \text{with } \mathrm{id}_{\mathcal{F}}(\mathcal{G}) = 0 \text{ and } \mathrm{Proj}(\mathcal{C}) \subseteq \mathcal{F} \cap \mathcal{G} \end{array} \right\}$$

along with the following subfamily of exact model structures:

$$\mathfrak{M}' := \left\{ \begin{array}{c} (\mathcal{S},\mathcal{M}) : \mathcal{S} \text{ is a thick sub-category of } \mathcal{C} \text{ and } \mathcal{M} = (\mathcal{Q},\mathcal{S},\mathcal{T}) \text{ is a projective exact} \\ \text{model structure on } \mathcal{S} \text{ such that } \mathcal{Q} \text{ is resolving in } \mathcal{C}, \text{ and } \mathcal{T} \subseteq \hat{\mathcal{Q}} \end{array} \right\}.$$

Proposition

Let $\mathcal{C}$ be an abelian category with enough projectives. Then there is a one-to-one correspondence:

$$\begin{array}{ccccc}
(\mathcal{X}, \omega) & (\mathcal{F}, \mathcal{F} \cap \mathcal{G}) & \mathrm{s}\mathfrak{F} & (\mathcal{X}, \omega) & (\mathcal{Q}, \mathcal{Q} \cap \mathcal{T}) \\
\Big\downarrow & \Big\uparrow & \Big\updownarrow & \Big\downarrow & \Big\uparrow \\
 & & \mathfrak{M}' & (\hat{\mathcal{X}}, \mathcal{M}^{\mathrm{proj}}_{\mathrm{AB}}(\mathcal{X}, \omega)) & (\mathcal{S}, \mathcal{M} = (\mathcal{Q}, \mathcal{S}, \mathcal{T})) \\
 & & \Big\updownarrow & & \\
(\mathcal{X}, \hat{\omega}) & (\mathcal{F}, \mathcal{G}) & \mathrm{s}\mathfrak{P} & &
\end{array}$$

Thank you for your attention!

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